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Full Length Research Paper

Analysis of a tandem queue with heterogeneous servers subject to catastrophes

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This study analyzed a tandem queueing system with blocking and no waiting line. The arrival process is assumed to be Poisson with rate λ. There are two non-identical servers in the system. The service times are exponentially distributed with parameters μ_1 and μ_2 at servers 1 and 2, respectively. Besides, the catastrophes occur in a Poisson manner with rate γ in the system. When server 1 is busy or blocked, the customer who arrives in the system leaves the system without being served. Such customers are called "lost customers". The probability of losing a customer was computed for the system.

Key words: Tandem queueing, heterogeneous servers, catastrophes, loss probability.

INTRODUCTION

The tandem queueing models with blocking used to model the systems encountered in fields such as manufacturing systems (Qiu and Zang, 2008; Seo and Lee, 2011), computer communications systems (Chu et al., 1981; Modiano et al., 1996), and wireless networks (Le et al., 2007; Niyato, 2010) have been examined and analyzed under the different assumptions suggested regarding arrival process, service process and service disciplines so far. The exact analytic solutions of such queueing models are only possible for the cases with a small number of servers (Gordon and Newell, 1967; Konheim and Reiser, 1976, 1978; Alpaslan, 1996; Grassmann and Drekic, 2000). Therefore, approximate solutions were obtained by using approximation methods for the systems of tandem queues with blocking in many studies in the literature (Boxma and Konheim, 1981; Foster and Perros, 1980; Altiok, 1982; Altiok and Stidham, 1982; Brandwajn and Jow, 1988; and Altiok, 1989).

Alpaslan (1996) dealt with the tandem queueing model with no waiting line and obtained the probability of lost customers in the system. Kumar et al. (2007) examined the *M*/*M*/2 queueing model with two parallel servers also considering the fact that catastrophes fitting the Poisson distribution with rate γ might take place. As soon as a catastrophe occurs, both servers are inactivated momentarily and, immediately afterwards, the system returns to its initial state with probability 1. Although tandem queueing models have been examined under many service disciplines in the studies in the literature, the case of catastrophe has not been considered in any studies. In real systems, a catastrophe might result either from outside the system/facility or from another service station. Computer networks with a virus infection might be considered examples of tandem queueing models with catastrophes. Furthermore, the return of the system to the initial state either automatically or by the admin due to the busy condition of the Internet systems or any hitch might be considered examples of models with catastrophes.

In this study, a system of tandem queue with heterogeneous servers subject to catastrophes is examined and the mean number of customers in the system and the probability of lost customers are computed. It is proved that the loss probability is

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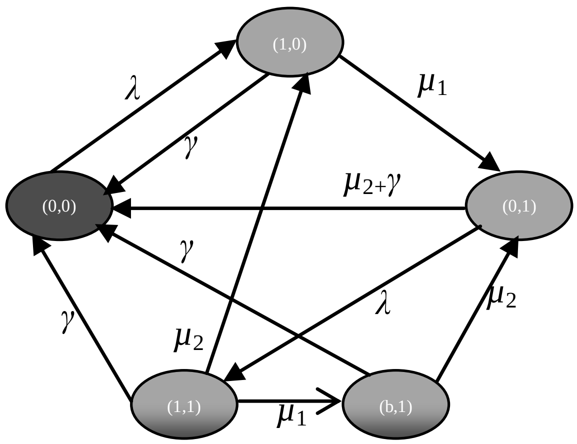

Figure 1. State transition-rate diagram.

minimum when the customer having arrived in the system is first served at the fast server and then at the slow server, respectively.

The results of this paper are organized as follows: the assumptions of the model are first described; followed by an analysis of the system with the Markov process and the steady-state probabilities of the system, the mean number of customers in the system and the loss probability are computed. Thereafter, the conditions under which the loss probability is optimum are determined. A numerical example of the model under consideration is subsequently provided, the obtained results are evaluated and the new studies likely to be made concerning the subject and recommendations are discussed.

THE MODEL AND ITS ASSUMPTIONS

"The tandem queueing system with blocking, heterogeneous servers and no waiting line subject to catastrophes" will be analyzed in this study. In this model, arrival times are Poisson distributed with parameter λ. There are two heterogeneous tandem servers in the system. Their mean service times are assumed to be different from each other. The service time of each customer at server *k* is random variable η_k and has an exponential distribution with parameter μ_k $(k=1,2)$, that is, $P(\eta_k \le t) = 1 - e^{-\mu_k t}$, $t > 0$. Apart from arrival and service processes, the catastrophes occur in a Poisson manner with rate γ in the system. Let the instants of catastrophe times be $\tau_0, \tau_1, \tau_2, \dots,$ where $0 = \tau_0 < \tau_1 < \cdots$ and $T = \tau_k - \tau_{k-1}$ for $k \ge 1$. Due to the relationship between Poisson distribution and exponential distribution, the time between two instants of catastrophe is exponentially distributed. That is, T has an exponential distribution with parameter γ, that is, $P(T \le t) = 1 - e^{-\gamma t}$, $t > 0$. As soon as a catastrophe takes place in the system, all customers are immediately destroyed. Both servers are inactivated momentarily and when there is a new arrival, both servers get ready to serve. Briefly, when there is a catastrophe in the system, the system returns to its initial state with probability 1.

Each customer arriving in the system is first served at server 1 and then at server 2, respectively. Waiting line is not allowed in front of the servers. If server 2 is busy when the service time has been completed at server 1, then server 1 is blocked until the service is completed at server 2. If server 1 is busy or blocked at the time of arrival of a customer in the system, that customer leaves the system without being served at all. Such customers are called "lost customers". Thus, the main problem herein is to compute the probability of lost customers in the system and minimize this probability. Loss probability and the minimization of loss probability were examined by Nath and Enns (1981), Yao (1987), Isguder and Uzunoglu-Kocer (2010), Isguder et al. (2011) and Isguder and Celikoglu (2012) particularly for queueing models with no waiting line.

ANALYZING THE MODEL USING MARKOV PROCESS

Let $\{X(t) = (u_t, v_t),\ t \ge 0\}$ be the number of customers in the system at time *t*, let u_t and v_t be the states of servers 1 and 2, respectively. Let $P_{00}(t) = P(X(t) = 0)$ be the probability that the system is empty at time *t*. Let $P_{10}(t) = P(X(t) = 1)$ be the probability that there is one customer served by server 1 in the system at time *t*. Let $P_{01}(t) = P(X(t) = 1)$ be the probability that there is one customer served by server 2 in the system at time *t*. Let $P_{11}(t) = P(X(t) = 2)$ be the probability that there are two customers in the system at time *t*, and that both servers are busy. In addition, let $P_{b1}(t) = P(X(t) = 1)$ be the probability that there is one customer who is served by server 2 in the system at time *t* and that server 1 is blocked. It is clear that $\{u_t, v_t\},\ t \ge 0$ is a continuous-parameter Markov process with state spaces $S = \{(0,0), (1,0), (0,1), (1,1), (b,1)\}$ and $P_{ij}(t) = P\{u_t = i, v_t = j\},\ (i,j) \in S$ (Figure 1).

Under the model assumptions, state probabilities $P_{ij}(t)$ satisfy the following system of differential equations:

$$\frac{dP_{00}(t)}{dt} = -\lambda P_{00}(t) + \mu_2 P_{01}(t) + \gamma[1 - P_{00}(t)], \qquad (1)$$

$$\frac{dP_{10}(t)}{dt} = -(\mu_1 + \gamma)P_{10}(t) + \lambda P_{00}(t) + \mu_2 P_{11}(t), \quad (2)$$

$$\frac{dP_{01}(t)}{dt} = -(\lambda + \mu_2 + \gamma)P_{01}(t) + \mu_1 P_{10}(t) + \mu_2 P_{b1}(t), \quad (3)$$

$$\frac{dP_{11}(t)}{dt} = -(\mu_1 + \mu_2 + \gamma)P_{11}(t) + \lambda P_{01}(t), \quad (4)$$

$$\frac{dP_{b1}(t)}{dt} = -(\mu_2 + \gamma)P_{b1}(t) + \mu_1 P_{11}(t). \quad (5)$$

The limits $P_{ij} = \lim_{t\to\infty} P\{u_t = i, v_t = j\}, (i, j) \in S$ exist and satisfy the system of linear equations. These equations are obtained from (1) to (5) on replacing the derivatives on the left by zero as follows:

$$\begin{aligned} &-\lambda P_{00} + \mu_2 P_{01} + \gamma - \gamma P_{00} = 0, \\ &-(\mu_1 + \gamma)P_{10} + \lambda P_{00} + \mu_2 P_{11} = 0, \\ &-(\lambda + \mu_2 + \gamma)P_{01} + \mu_1 P_{10} + \mu_2 P_{b1} = 0, \\ &-(\mu_1 + \mu_2 + \gamma)P_{11} + \lambda P_{01} = 0, \\ &-(\mu_2 + \gamma)P_{b1} + \mu_1 P_{11} = 0. \end{aligned} \quad (6)$$

Steady-state probabilities P_{ij} are expressed in terms of P_{00}

$$P_{01} = \frac{(\lambda + \gamma)P_{00} - \gamma}{\mu_2}, \quad (7)$$

$$P_{10} = \frac{\lambda[(\lambda + \mu + 2\gamma)P_{00} - \gamma]}{(\mu_1 + \gamma)(\mu + \gamma)}, \quad (8)$$

$$P_{11} = \frac{\lambda[(\lambda + \gamma)P_{00} - \gamma]}{\mu_2(\mu + \gamma)}, \quad (9)$$

$$P_{b1} = \frac{\lambda\mu_1[(\lambda + \gamma)P_{00} - \gamma\}}{\mu_2(\mu_2 + \gamma)(\mu + \gamma)}, \quad (10)$$

where $\mu = \mu_1 + \mu_2$. Unknown probability P_{00} is determined using the condition $P_{00} + P_{10} + P_{01} + P_{11} + P_{b1} = 1$ as follows:

$$P_{00} = \frac{\mu_2 + \gamma + \frac{\lambda\gamma}{\mu + \gamma}\left(\frac{\mu_1}{\mu_2 + \gamma} + \frac{\mu + \gamma}{\mu_1 + \gamma}\right)}{\lambda + \gamma + \frac{\mu_2(\lambda + \mu_1 + \gamma)}{\mu_1 + \gamma} + \frac{\lambda(\lambda + \gamma)}{\mu + \gamma}\left(\frac{\mu_1}{\mu_2 + \gamma} + \frac{\mu + \gamma}{\mu_1 + \gamma}\right)} \quad (11)$$

where $\mu = \mu_1 + \mu_2$. By using Equations (7), (8), (9), (10) and (11), the expected number of customers in the system, L_s, and the probability of lost customers in the system, P_L, are obtained as follows:

$$\begin{aligned} L_s &= P_{01} + P_{10} + 2(P_{11} + P_{b1}) \\ &= 1 - P_{00} + P_{11} + P_{b1} \\ &= 1 + \frac{[\lambda(\lambda + \gamma) - \mu_2(\mu_2 + \gamma)]P_{00} - \gamma}{\mu_2(\mu_2 + \gamma)} \end{aligned} \quad (12)$$

$$\begin{aligned} P_L &= P_{10} + P_{11} + P_{b1} \\ &= 1 - P_{00} - P_{01} \\ &= \frac{\mu_2 + \gamma - (\lambda + \mu_2 + \gamma)P_{00}}{\mu_2}. \end{aligned} \quad (13)$$

OPTIMIZATION OF LOSS PROBABILITY

As no waiting line is available in the system, some of the customers arriving in the system have to leave the system without being served. Such customers are called "lost customers". The probability of lost customers is easily obtained from formula (13). In such systems, the minimization of loss probability is a serious problem. In the model considered in this study, the customer arriving in the system first enters the first server and then the second server, respectively. How should these servers be put in order in terms of the minimization of loss probability? The answer to this question is revealed with a theorem.

Theorem 1

Loss probability P_L takes the minimum value when servers are ordered according to the values of μ_1 and μ_2, satisfying inequality $\mu_1 \geq \mu_2$, under the condition $\mu_1 + \mu_2 = \mu$.

Proof

Let $\tilde{P}_L$ be the probability of lost customers in the system

Table 1. Calculations of the performance measures using the given values of parameters.

Parameter	Value	Parameter	Value
λ	1.5600	$\boldsymbol{\lambda}$	1.5600
μ_1	2.1300	$\boldsymbol{\mu_1}$	5.9200
μ_2	5.9200	$\boldsymbol{\mu_2}$	2.1300
γ	0.1000	$\boldsymbol{\gamma}$	0.1000
Performance measure	**Calculation**	**Performance measure**	**Calculation**
P_{00}	0.4108	$\boldsymbol{P_{00}}$	0.4692
L_s	0.5621	$\boldsymbol{L_s}$	0.7921
P_L	0.4162	$\boldsymbol{P_L}$	0.3161

for $\mu_1 \le \mu_2$ under the condition and $\mu_1 + \mu_2 = \mu$.

For easiness, let us reorganize the loss probability provided with formula (13) for $\gamma = 0$ in the following way:

$$P_L = \frac{\lambda\mu^2\mu_2 + \lambda^2\mu\mu_2 + \lambda^2\mu_1^2 - \lambda\mu\mu_1\mu_2}{\lambda\mu^2\mu_2 + \lambda^2\mu\mu_2 + \lambda^2\mu_1^2 + \mu\mu_1\mu_2^2}, \qquad (14)$$

If we multiply both the numerator and denominator of formula (14) by μ_1, formula (15) is obtained as follows:

$$P_L = \frac{(\lambda\mu^2\mu_1\mu_2 + \lambda^2\mu\mu_1\mu_2) + \lambda^2\mu_1^3 - \lambda\mu\mu_1^2\mu_2}{(\lambda\mu^2\mu_1\mu_2 + \lambda^2\mu\mu_1\mu_2) + \lambda^2\mu_1^3 + \mu\mu_1^2\mu_2^2}, \qquad (15)$$

For easiness, let θ symbolize term $(\lambda\mu^2\mu_1\mu_2 + \lambda^2\mu\mu_1\mu_2)$ in formula (15). In other words, considering that $\mu = \mu_1 + \mu_2$, the formula of loss probability is obtained as follows:

$$P_L = \frac{\theta + \lambda\mu_1^3(\lambda - \mu_2) - \lambda\mu_1^2\mu_2^2}{\theta + \lambda^2\mu_1^3 + \mu\mu_1^2\mu_2^2}, \qquad (16)$$

Term θ is present both in the numerator and denominator of formula (16) and it takes the same value for $\mu_1 \ge \mu_2$ and $\mu_1 \le \mu_2$, respectively. Terms $\lambda\mu_1^2\mu_2^2$ and $\mu\mu_1^2\mu_2^2$ are present in the numerator and denominator of Equation (16), respectively. Both terms take the same value for $\mu_1 \ge \mu_2$ and $\mu_1 \le \mu_2$, respectively. The value that term $\lambda^2\mu_1^3$ in the denominator of Equation (16) takes for $\mu_1 \ge \mu_2$ is either equal to or greater than the value it takes for $\mu_1 \le \mu_2$.

Hence, the value the denominator of Equation (16) takes for $\mu_1 \ge \mu_2$ is either equal to or greater than the value it takes for $\mu_1 \le \mu_2$ under the condition $\mu_1 + \mu_2 = \mu$.

Let the numerator of Equation (16) be examined. It will be $(\lambda - \mu_2) > 0$, $(\lambda - \mu_2) < 0$ and $(\lambda - \mu_2) = 0$ for $\lambda > \mu_2$, $\lambda < \mu_2$ and $\lambda = \mu_2$, respectively. For $(\lambda - \mu_2) < 0$ and $(\lambda - \mu_2) = 0$, the denominator of Equation (16) always decreases under the condition $\mu_1 + \mu_2 = \mu$ while μ_1 is increasing and μ_2 is decreasing. Depending on the above-mentioned remarks, the value that the numerator of Equation (16), $\theta + \lambda\mu_1^3(\lambda - \mu_2) - \lambda\mu_1^2\mu_2^2$, takes for $\mu_1 \ge \mu_2$ is either equal to or smaller than the value it takes for $\mu_1 \le \mu_2$ under the conditions $(\lambda - \mu_2) < 0$, $(\lambda - \mu_2) = 0$ and $\mu_1 + \mu_2 = \mu$. *On the other hand,* the numerator and the denominator of Equation (16) should be considered together for $(\lambda - \mu_2) > 0$. In this case, the increase in the denominator of Equation (16) is either equal to or greater than the increase in its numerator while μ_1 is increasing and μ_2 is decreasing. $\lambda^2\mu_1^3 + \mu\mu_1^2\mu_2^2 \ge \lambda\mu_1^3(\lambda - \mu_2) - \lambda\mu_1^2\mu_2^2$ under the condition $\mu_1 + \mu_2 = \mu$.

In conclusion, loss probability P_L obtained for $\mu_1 \ge \mu_2$ is either equal to or smaller than the loss probability obtained for $\mu_1 \le \mu_2$ under the condition $\mu_1 + \mu_2 = \mu$. That is, $P_L \le \tilde{P}_L$. The proof has been completed.

When the customers are first served at the fast server and then at the slow server, the system becomes optimal in terms of the probability of customers leaving without being served.

NUMERICAL EXAMPLE

In this part of the work, the probability of lost customers in the system and the mean number of customers in the system are explained with a numerical example. It is numerically shown that the loss probability is minimum when the customers who have arrived in the system are first served at the fast server and then at the slow server. The assumptions of the system are as explained in the second section. The parameters of the system, the mean number of customers computed by the help of these parameters and the loss probability are as in Table 1.

The P_{00}, L_s and P_L values obtained in Table 1 were computed by the help of formulae (11), (12) and (13), respectively. As clearly seen from this table, the loss probability (P_L) takes its minimum value if the customer who has arrived in the system is first served at the fast server and then at the slow server. On the other hand, the mean number of customers in the system (L_s) increases as the loss probability in the system decreases.

DISCUSSION

In this study, "the tandem queueing system with blocking and two heterogeneous servers" was analyzed using the Markov process. The mean number of customers in the system and the probability of lost customers were obtained. Unlike the studies in the literature, some modeling that is more approximate to real systems was made by considering the reality that catastrophes might occur. Furthermore, considering that the customer having arrived in the system was first served at the fast server and then at the slow server, it was proved that the probability of lost customers in the system was minimum. The obtained results were supported with a numerical example.

On the other hand, the absence of waiting lines in the system is the weakness of the model. The addition of a limited or unlimited number of waiting lines to the system by further generalizing this model will enable the model to be more realistic. The computation of the probability of lost customers in the system especially by adding a limited number of waiting lines to the model will enable the model to be implemented in more fields in real life.

Moreover, the mathematical model established in this study and the obtained results will facilitate the modeling of more complicated systems with simulation and the computation of performance measures such as the mean number of customers in the system and the loss probability with the simulation approach and they can be used to test the accuracy of the results obtained with the simulation approach.

RECOMMENDATIONS FOR FUTURE RESEARCH

The results obtained for the tandem queueing model considered in this paper will guide the analysis of complex models which are more approximate to real systems. Analysis of the model to be obtained by adding a limited or unlimited number of waiting lines to this model under consideration can be thought as future research. On the other hand, mathematical modeling of the system will become more difficult when the number of servers is more than two and it will be impossible to obtain the exact solutions of the performance measures of the system. In such cases, approximate solutions might be obtained by developing simulation techniques or approximate methods. In terms of the minimization of lost customers, whether the optimization performed considering the entries first into the fast server and then into the slow server is preserved when the number of servers is three or more can be proved by a simulation optimization.

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