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Systematic generation of network functions for linear circuits using modified nodal approach

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In this paper, a systematic and efficient formulation method is presented for obtaining the network functions of linear or linearized time-invariant circuits. The method employs the modified nodal approach to obtain the system equations. The technique is based on developing a matrix formulation for generating network functions. By using both symbolic manipulation of algebraic expressions and numeric processes, the network functions are expressed with a matrix-based method. Application examples are given to illustrate features of the method.

Key words: Network functions, modified nodal approach, formulation method.

INTRODUCTION

Network functions are used as an effective tool in the analysis and design of electrical circuits. Many circuit characteristics such as voltage/current gains, input/output impedances, poles/zeros of circuits can be computed from network functions. Therefore, network functions can be a powerful tool for designers of analog integrated circuits. Several approaches used in obtaining network functions are given in symbolic or numerical format. Some symbolic methods about network function generation are proposed by Aguirre and Carlosena (2000), Djordjevic and Petkovic (2004), Pierzchala and Rodanski (2001), Ruzhang et al. (1995), Shi and Tan (2001), Topa and Simion (1996), Yu and Sechen (1996) and Wambacq et al. (1996). Nedelea et al. (2003) propose the computer-aided network function approximation for analog low and high pass filters. Yuan (2000) investigates the periodicity of network functions of linear periodically time-varying systems. Applications about the realization of transfer functions, one of the main components of network functions, are given by Sagbas et al. (2010), Psychalinos (2007) and Raut (2006).

In this paper, the algebraic method for obtaining the network functions of linear or linearized time-invariant circuits is proposed. For setting up the circuit equations, the modified nodal approach (MNA), which is one of the most popular methods of circuit analysis, is used. The state variables method, the other popular method and based on the graph theoretical approach, were developed before the modified nodal analysis. It involves intensive mathematical process and has major limitations in the formulation of circuit equations. Some of these limitations arise because the state variables are capacitor voltages and inductor currents.

Every circuit element cannot be easily included into the state equations. Because of the drawbacks of state variables analysis, the modified nodal analysis was first introduced by Ho et al. (1975) and has been further developed by including several circuit elements (transformer, semiconductor devices, short circuit, etc.) into the system equations (Vlach and Singhal, 1983; Thomas and Rosa, 2006; Nilsson and Riedel, 2005; Yildiz, 2006). In this method, the system equations can be also obtained by inspection. It allows circuit equations to be easily and systematically obtained without any limitation. This method is used for circuit synthesis of passive descriptor systems (Reis, 2010) and for computing the smallest, the largest and a given subset of the largest eigenvalues associated with linear time-invariant circuits (Exposito et al., 2009).

In this paper, how to use the advantages of modified nodal approach in obtaining the network functions of linear circuits is shown. The main contribution of the paper is that it gives a systematic formulation method in terms of variables of MNA. The network functions can be obtained as both symbolic and numeric with the proposed method.
Figure 1. System model.

System equations

The modified nodal equations and the output equations of any circuit are given in t-domain (Equations (1) and (2)) and in s-domain, (Equations (3) and (4)). In this method, the equations are first obtained in s-domain. Later, during analysis, they are transformed into t-domain or frequency domain. Since the network functions are expressed in s-domain, the system equations will be examined in s-domain. The nodal and output equations together are called the system model (Figure 1):

\[
\begin{align*}
Gx(t) + C \frac{dx}{dt} &= Bu(t) \\
y(t) &= Tx(t) \\
GX(s) + sCX(s) &= BU(s) \\
Y(s) &= TX(s)
\end{align*}
\]

(1) \hspace{2cm} (2) \hspace{2cm} (3) \hspace{2cm} (4)

Where: G, C, B, T are coefficient matrices. All conductances and frequency-independent values arising in the MNA formulation are stored in matrix G, capacitor and inductor values, which are frequency-dependent in matrix C. U(s) represents the input (voltage or current source), Y(s) represents the output variable (voltage/current). The unknown vector X(s) contains both voltage and current variables. MNA can handle all types of active and passive elements. It is a very important property of MNA.

Taking into account the types of variables, the unknown vector is partitioned as follows:

\[
X(s) = \begin{bmatrix} X_1(s) \\ \vdots \\ X_{n-1}(s) \end{bmatrix}
\]

(5)

Here, \(X_1(s)\) represents nodal voltage variables, \(X_2(s)\) represents current variables relating to independent and controlled voltage sources, inductors, short circuit elements, etc. If there are n nodes and m current variables in a circuit, \(X_1(s)\) vector contains \(n-1\) nodal voltage variables except reference node (ground) and \(X_2(s)\) vector contains m current variables. Thus, the unknown vector \(X(s)\) contains \(k = n - 1 + m\) variables:

\[
X_1(s) = \begin{bmatrix} U_1(s) \\ U_2(s) \\ \vdots \\ U_{n-1}(s) \end{bmatrix}, \quad X_2(s) = \begin{bmatrix} I_1(s) \\ I_2(s) \\ \vdots \\ I_m(s) \end{bmatrix}
\]

(6)

From Equation (3), \(X(s)\) is obtained as follows:

\[
X(s) = \left[ G + sC \right]^{-1} BU(s) = W(s)U(s)
\]

(7)

The \(W(s)\) vector is of order \((n - 1 + m) \times 1\). It is created from coefficient matrices of system equations. Let’s consider Equations (7) and (8) together:

\[
X(s) = \begin{bmatrix} U_1(s) \\ \vdots \\ U_{n-1}(s) \end{bmatrix} = W(s)U(s) = \begin{bmatrix} W_1(s) \\ \vdots \\ W_{n-1}(s) \end{bmatrix}
\]

(9)

Equation (9) is also expressed separately as follows:

\[
\begin{align*}
U_1(s) &= W_1(s)U(s) \\
U_2(s) &= W_2(s)U(s) \\
\vdots \\
U_{n-1}(s) &= W_{n-1}(s)U(s) \\
I_1(s) &= W_n(s)U(s) \\
\vdots \\
I_m(s) &= W_k(s)U(s)
\end{align*}
\]

(10)
The elements of $X(s)$ vector in Equation (9) or (10) are expressed in terms of the elements of $W(s)$ vector and the input.

In the two port circuit represented in Figure 2, the nodes of input port are 1, $n$ and the nodes of output port are $n-1$, $n-2$. Node $n$ is always taken as a reference node (ground, $U_n = 0$). Sometimes, the input and the output ports are connected to a common reference node, as in the illustrative examples. The voltage source ($U_i(s)$) in Figure 2 is a symbolic source to be used in obtaining the network functions.

In Figure 2, $U_i(s) = U_{i1}(s)$ and $U_o(s) = U_{n-2}(s) - U_{n-1}(s)$. That is, the input voltage of circuit ($U_i$) is always equal to the first nodal voltage (the first variable of system) in $X_i(s)$ vector, the output voltage of circuit ($U_o$) is always equal to the difference between the last two nodal voltages (the $(n-2)$th and $(n-1)$th variables of system) in $X_i(s)$ vector. In Equation (9) or (10), $W_i(s) = 1$ base on the fact that $U(s) = U_i(s) = U_{i1}(s)$. The input current ($I_i(s)$) is the source current. It is located in the last row of $X_i(s)$ or $X(s)$ vector. That is, $I_i(s) = I_m(s)$. For the output current ($I_o(s)$), the output port in Figure 2 must be terminated with any circuit element. For instance, the circuit in Figure 2 is terminated with a resistor.

In order to obtain the system equations as an input, a symbolic current source ($J_i(s)$) can be also used. In this case, the structure of equations is similar to Equation (9) or (10). Of course, the elements of $W(s)$ vector are different for every input ($U_i(s)$ or $J_i(s)$).

The above expressions are given for the circuit having one input and one output in Figure 2. For the case of $p$ inputs and $q$ outputs, as in the circuit in Figure 3, the expressions can be generalized. In this case, the $W(s)$ vector is of order $(n-1+m) \times p$.

$$X(s) = W(s)U(s) = \begin{bmatrix} W_{i1}(s) & \cdots & W_{ip}(s) \\ W_{21}(s) & \cdots & W_{2p}(s) \\ \vdots & \ddots & \vdots \\ W_{k1}(s) & \cdots & W_{kp}(s) \end{bmatrix} \begin{bmatrix} U_{i1}(s) \\ \vdots \\ U_{ip}(s) \end{bmatrix}$$

(11)

In this paper, for one-input and one-output circuits, as in
Figure 2, the network functions are expressed. For multi-input and multi-output circuits, similar expressions are valid.

Network functions

Using Equation (9) or (10), network functions and various domain responses relating to any circuit can be expressed systematically in terms of the elements of \( W(s) \) vector.

Transfer functions

The transfer function \( (H(s)) \) is defined as the ratio of the output response to the input. It can be obtained by using Equations (3) and (4).

From Equation (3):

\[
X(s) = [G + sC]^{-1} BU(s) = W(s)U(s) \quad (12)
\]

The output equation is given as:

\[
Y(s) = TX(s) = T[G + sC]^{-1} BU(s) \quad Y(s) = TW(s)U(s) \quad (13)
\]

From Equation (13), the transfer function is expressed in terms of the matrices of MNA system:

\[
H(s) = \frac{Y(s)}{U(s)} = T[G + sC]^{-1} B = TW(s) \quad (14)
\]

Equation (14) is the general statement of transfer functions. There are four kinds of transfer functions according to input and output variables.

1. Voltage transfer function:

\[
H_1(s) = \frac{U_o(s)}{U_i(s)} \quad (15a)
\]

2. Current transfer function:

\[
H_2(s) = \frac{I_o(s)}{I_i(s)} \quad (15b)
\]

3. Transfer impedance function:

\[
H_3(s) = \frac{U_o(s)}{I_i(s)} \quad (15c)
\]

4. Transfer admittance function:

\[
H_4(s) = \frac{I_o(s)}{U_i(s)} \quad (15d)
\]

For obtaining the transfer functions, the voltage and current variables relating to input and output ports are expressed in terms of the elements of \( W(s) \) vector created and the source, according to Figure 2 and Equation (10). For the current transfer function (Equation 15b) and transfer admittance function (Equation 15d), the output port in Figure 2 must be terminated with any circuit element. Here, the expressions are given for the case terminated with a resistor, as in Figure 2. But, the method is general and can be applied for any circuit element:

i) Voltage transfer function:

\[
H_{1i} = \frac{U_o(s)}{U_i(s)} = \frac{U_{\alpha 2}(s) - U_{\alpha 1}(s)}{U_{\alpha 1}(s)} = \frac{[W_{\alpha 2}(s) - W_{\alpha 1}(s)]U_{\beta 1}(s)}{W_{\alpha 1}(s)U_{\beta 1}(s)} = \frac{W_{\alpha 2}(s) - W_{\alpha 1}(s)}{W_{\alpha 1}(s)} \quad (16)
\]

ii) Current transfer function:

\[
H_{2i} = \frac{I_o(s)}{I_i(s)} = \frac{U_{\alpha 2}(s) - U_{\alpha 1}(s)}{I_{\alpha 1}(s)} = \frac{[W_{\alpha 2}(s) - W_{\alpha 1}(s)]U_{\beta 1}(s)}{W_{\alpha 1}(s)U_{\beta 1}(s)} = \frac{W_{\alpha 2}(s) - W_{\alpha 1}(s)}{W_{\alpha 1}(s)} \quad (17)
\]

iii) Transfer impedance function:

\[
H_{3i} = \frac{U_o(s)}{I_i(s)} = \frac{U_{\alpha 2}(s) - U_{\alpha 1}(s)}{I_{\alpha 1}(s)} = \frac{[W_{\alpha 2}(s) - W_{\alpha 1}(s)]U_{\beta 1}(s)}{W_{\alpha 1}(s)U_{\beta 1}(s)} = \frac{W_{\alpha 2}(s) - W_{\alpha 1}(s)}{W_{\alpha 1}(s)} \quad (18)
\]

iv) Transfer admittance function:

\[
H_{4i} = \frac{I_o(s)}{U_i(s)} = \frac{U_{\alpha 2}(s) - U_{\alpha 1}(s)}{I_{\alpha 1}(s)} = \frac{[W_{\alpha 2}(s) - W_{\alpha 1}(s)]U_{\beta 1}(s)}{W_{\alpha 1}(s)U_{\beta 1}(s)} = \frac{W_{\alpha 2}(s) - W_{\alpha 1}(s)}{W_{\alpha 1}(s)} \quad (19)
\]

Input Impedance

The input impedance or driving-point impedance, \( Z(s) \), is expressed according as shown in Figure 2.

\[
Z(s) = \frac{U_i(s)}{I_i(s)} \quad (20)
\]

The impedance relates the voltage and the current at input terminals. \( U_i(s) \) is the driving source and \( I_i(s) \) is the source current in Figure 2. The source current: \( I_i(s) = I_m(s) \):

\[
Z(s) = \frac{U_i(s)}{I_i(s)} = \frac{U_i(s)}{I_m(s)} = \frac{U_i(s)}{W_k(s)U_i(s)} = \frac{1}{W_k(s)} \quad (21)
\]
Frequency-domain response

For frequency response of system in Equations (12), (13), (16)-(19), and (21), respectively, s was replaced by \( i\omega \):

\[
X(i\omega) = \left[ G + i\omega C \right]^{-1} BU(i\omega) \tag{22a}
\]

\[
Y(i\omega) = T\left[ G + i\omega C \right]^{-1} BU(i\omega) \tag{22b}
\]

\[
H_1(i\omega) = \frac{U_0(i\omega)}{U_i(i\omega)} = \frac{W_{n-2}(i\omega) - W_{n-1}(i\omega)}{W_1(i\omega)} \tag{22c}
\]

\[
H_2(i\omega) = \frac{I_0(i\omega)}{I_i(i\omega)} = \frac{W_{n-2}(i\omega) - W_{n-1}(i\omega)}{W_k(i\omega)R} \tag{22d}
\]

\[
H_3(i\omega) = \frac{U_0(i\omega)}{I_i(i\omega)} = \frac{W_{n-2}(i\omega) - W_{n-1}(i\omega)}{W_k(i\omega)} \tag{22e}
\]

\[
H_4(i\omega) = \frac{I_0(i\omega)}{U_i(i\omega)} = \frac{W_{n-2}(i\omega) - W_{n-1}(i\omega)}{W_j(i\omega)R} \tag{22f}
\]

\[
Z(i\omega) = \frac{1}{W_k(i\omega)} \tag{22g}
\]

Illustrative examples

Two examples to show the proposed method relating to obtaining the network functions were given.

Example 1

Consider the circuit in Figure 4. The system equations, the transfer functions \( U_o/U_i, U_o/I_i \), and the input impedance of the circuit will be obtained. Element values are \( R = 2\Omega, C_1 = C_2 = 3F, L = 5H \).

The input and output of the circuit has a common reference node (4). The circuit has \( n - 1 = 3 \) nonreference nodes. Thus, in the MNA system, \( X_1(s) \) vector contains 3 nodal voltage variables. The current variables in \( X_2(s) \) vector are \( I_L, I_i \). Thus, in the circuit, \( k = n - 1 + m = 5 \). The representation of voltage source \( U_i \), as an input, is used to obtain the network functions.

Nodal (main) equations in s-domain are as follows:

\[
1 \rightarrow \ G(U_1 - U_2) - I_1 = 0
\]

\[
2 \rightarrow - G(U_1 - U_2) + sC_1U_2 + I_L = 0
\]

\[
3 \rightarrow sC_2U_3 - I_L = 0
\]

Additional equations:

\[
U_1 = U_i, \quad U_2 - U_3 = sLI_L
\]

The overall equations constitute the MNA system (Equation 23). The output equation of system is given in Equation (24). The system model containing both MNA equations and output equation can be given in matrix form, as in Figure 1.

\[
\begin{bmatrix}
G + sC \end{bmatrix}X(s) = BU(s) \quad \rightarrow \quad GX(s) + sCX(s) = BU(s)
\]

\[
\begin{bmatrix}
G & -G & 0 & 0 & -I \\
-G & G & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & -L
\end{bmatrix}
\begin{bmatrix}
U_1 \\
U_2 \\
U_3 \\
I_L \\
I_i
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\left[ \begin{bmatrix}
G & -G & 0 & 0 & -I \\
-G & G & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & -L
\end{bmatrix}
\begin{bmatrix}
U_1 \\
U_2 \\
U_3 \\
I_L \\
I_i
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\right]
\]
The desired transfer functions are obtained as follows:

Transfer functions:

\[ H_i(s) = \frac{U_i(s)}{U_i(s)} = \frac{W_i(s)}{W_i(s)} = \frac{W_i(s)}{W_i(s)} \]

As seen from Equation (26), because \( U_i(s) \) is equal to \( W_i(s) = 1 \).

Transfer functions:

The desired transfer functions are obtained as follows:

(a) \( U_p/U_i \)

(b) \( U_p/I_i \)

\[ H_i(s) = \frac{U_i(s)}{I_i(s)} = \frac{W_i(s)/U_i(s)}{W_i(s)/U_i(s)} = \frac{W_i(s)/U_i(s)}{W_i(s)/U_i(s)} \]

Input impedance:

\[ Z(s) = \frac{U_i(s)}{I_i(s)} = \frac{W_i(s)/U_i(s)}{W_i(s)/U_i(s)} = \frac{s^3RLC + s^2LC + s(\text{RC} + \text{RC}) + 1}{s(s^2LC + \text{C} + \text{C})} \]

Example 2

Consider the circuit in Figure 5. In the system equations, the voltage transfer function \( (U_p/U_i) \), and the input impedance of the circuit will be obtained. Element values are \( R_1 = 4\Omega, R_2 = 5\Omega, C_1 = 1\text{F}, C_2 = 2\text{F}. \)

Node \( n \) is chosen as a reference, \( U_n = 0 \). The voltage and current constraints of ideal Op-Amp are \( I_p = 0, I_n = 0, U_p - U_n = 0 \). In the MNA system, \( X(s) \) vector contains 3 nodal voltage variables \( (U_a, U_b, U_c) \). The current variable in \( X(s) \) vector is \( I \). Thus, in the circuit, \( k = n - 1 + m = 4 \).

The representation of voltage source \( (U) \), as an input, is used to obtain the network function and the input impedance.

Nodal (main) equations in s-domain are given as:

\[ a \rightarrow G_1(U_a - U_b) + I_i = 0 \]
\[ b \rightarrow -G_1(U_a - U_b) + sC_2(U_b - U_n) + sC_1(U_b - U_c) = 0 \]
\[ n \rightarrow -sC_2(U_b - U_n) + G_2(U_n - U_c) + I_n = 0 \]

Additional equations:

\[ U_a = U_i, \quad U_p - U_n = 0 \]
The overall equations constitute the MNA system (Equation 30). The output equation of system is given in Equation (31). The system model containing both MNA equations and output equation can be given in matrix form, as in Figure 1.

\[
\begin{bmatrix}
G + sC \\
G - G_0 - G_1 \\
G_2 \\
1
\end{bmatrix} \begin{bmatrix}
X(s) \\
U_a(s) \\
U_b(s) \\
I(s)
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} \begin{bmatrix}
U_a(s) \\
U_b(s) \\
U_c(s) \\
I(s)
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
U_a(s) \\
U_b(s) \\
U_c(s) \\
I(s)
\end{bmatrix}
\]

The output equation:

\[
Y(s) = TX(s)
\]

By using this system model, the vector \( W(s) \) is created. Thus, the desired network function and the input impedance are systematically calculated.

\[
X(s) = \begin{bmatrix}
U_a(s) \\
U_b(s) \\
U_c(s) \\
I(s)
\end{bmatrix} = (G + sC)^{-1} BU_a(s) = W(s)U_a(s) = \begin{bmatrix}
W_1(s) \\
W_2(s) \\
W_3(s) \\
W_4(s)
\end{bmatrix}
\]

Where,

\[
\begin{bmatrix}
W_1(s) \\
W_2(s) \\
W_3(s) \\
W_4(s)
\end{bmatrix} = \begin{bmatrix}
1 \\
\frac{1}{1 + sRC + sR_1C_1 + s^2RRCC} \\
-\frac{sRC_2}{1 + sRC + s^2RRCC} \\
\frac{1 + sRC_1 + s^2RRCC}{1 + sRC + s^2RRCC}
\end{bmatrix}
\]

Voltage transfer function:

\[
H_1(s) = \frac{U_0(s)}{U_1(s)} = \frac{U_c(s)}{U_a(s)} = \frac{W_3(s)U_a(s)}{W_2(s)U_a(s)} = \frac{W_3(s)}{W_2(s)}
\]

\[
H_2(s) = \frac{U_0(s)}{U_1(s)} = \frac{-sRC_2}{s^2R_1R_2C_1C_2 + s(R_1C_1 + R_2C_2) + 1}
\]

Input impedance:

\[
H_3(s) = \frac{U_0(s)}{U_1(s)} = \frac{-10s}{40s^2 + 12s + 1}
\]

Conclusion

The paper introduces a systematic matrix-based representation for the network functions of linear or linearized time-invariant circuits. The proposed method is based on the modified nodal approach, suitable for computer-aided analysis of active and passive circuits, and creating a matrix formulation. The main contribution of the paper is that it gives a systematic formulation method in terms of the components of the created vector, \( W(s) \). Transfer functions and input impedance relating to the examples show the efficiency of the approach.

The system equations and network functions can be obtained systematically by inspection. Therefore, for future work, a computer program about the network functions and frequency domain analysis of active and passive circuits can be written by using the presented method. Moreover, the noise analysis, one of the interesting applications of network analysis, can be also realized by this method.

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