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Full Length Research Paper

Investigation of gradually varied flows using differential quadrature method

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In this study, the water surface elevations in gradually varied flows are calculated by Differential Quadrature Method (DQM) which is recently used in Hydraulics Engineering. Water surface profiles can be calculated by solving the systems of equations that are linear at uniform channel, nonlinear at non-uniform channel. A uniform channel is selected as on illustrative example. The obtained results are compared with those of Direct Step Method. Gradually varied flow is calculated for M and S profiles in a trapezoidal channel. The differences between the results are negligibly small.

Key words: Gradually varied flow, water surface profile, differential quadrature method.

INTRODUCTION

As the differential equation of water surface written for Gradually Varied Flows (GVF) cannot be integrated directly, various numerical methods are used. Calculation of gradually varied flows with free surface is an important subject in hydraulic engineering. Numerous discussions have taken place so far on the subject of direct integration of differential equation of water surface in uniform channels with simple geometric shapes. Bresse (1860) realized direct integration for wide rectangular channels using Chezy equation. Later and Masoni (1900) proposed an approximate direct integration method for channels with rectangular cross sections. Bakhmeteff (1932) proposed a direct integration method that can be applied to all cross sections. Chow (1955) gives an equation for direct integration. Kumar (1978) gives a direct integration method for rectangular and triangle cross sections. Patil et al. (2001) improved the integral method proposed by Chow (1955). Venutelli (2004) realized the integration of differential equation of water surface for wide rectangular channels using Manning equation.

For the purpose of making calculation in desired cross sections for non-uniform channels, Standard Step Method is used. Henderson (1966) and French (1985) reported that Newton-Raphson approximation is suitable for the infinitely wide channels. Paine (1992) and Rhodes (1995) applied the same scheme for trapezoidal and general cross-sectional channels, respectively. The above analyses for the application of Newton-Raphson approximation were based on the Manning equation of flow resistance. Rhodes (1998) included the flow resistance equations Chezy and Colebrook-White in the analysis of Gradually Varied Flows. Dey (2000) made use of Chebyshev approach in the solution of the equation using Standard Step Method. Direct Step Method is use in uniform cross section channels.

Differential Quadrature Method (DQM) used in this study was developed by Richard Bellman (1971, 1972). This method proposes solution for the equations of any system obtained in differential form by including the present boundary-initial conditions into the equation. Using DQM, Shu and Richards (1992) made studies on some fields such as fluid mechanics, sprain and bending of plates and beams. In some recent studies, (Fung, 2001) principles of differential quadrature method were applied for the solution of the initial value problems encountered in the fields of heat transfer. Civalek (2003), examined free and forced vibrations with degrees of single and multiple freedom using harmonic differential quadrature method. In Shu et al. (2003) local radial basis function-based differential quadrature method was presented. In Shu et al. (2004) the differential quadrature (DQ) method was used to simulate the eccentric Couette–Taylor vortex flow in an annulus between two eccentric cylinders with rotating inner cylinder and stationary outer cylinder. Natural convection in a differentially heated cubic enclosure is studied by solving the velocity–vorticity form of the Navier–Stokes equations by a generalized differential quadrature (GDQ) method (Lo et al., 2005). The local multi-quadric differential quadrature (LMQDQ) method is applied on three-dimensional incompressible flow problems (Ding et al., 2006). DQM has been used in recent years in hydraulics engineering. The method has been used for unsteady open channel flows (Hashemi et al., 2006, 2007; Kaya et al., 2010). Water surface profile in subterranean channel by differential quadrature method (DQM) has been modeled by Robati and Barani (2009).

In order to obtain satisfactory results in the analyses of GVF problems using Standard Step Method and Direct Step Method, number of grid points must be increased. However, fewer number of grid points are required while using DQM. It has been noticed in previous studies that DQM results are more harmonious with those of the analytical results when compared with other numerical solutions.

MATERIALS AND METHODS

Gradually varied flow

In gradually varied flows, differential equation of water surface is written as follows

$$\frac{dy}{dx} = \frac{So - Sf}{1 - Fr^2} \tag{1}$$

where , y is the water depth, x is the space in the direction of the flow, So is the slope of the channel bottom; Sf is the slope of energy line; and Fr is the Froude number (Chow, 1955). In a channel with any geometric shape, Froude number is written as follows:

$$Fr^2 = \frac{Q^2 B}{gA^3} \tag{2}$$

For Sf such relations as Chezy equation (Equation 3) and Manning equation (Equation 4)

$$Sf = \left[\frac{Q}{CA}\right]^2 \frac{1}{R} \tag{3}$$

$$Sf = \left[\frac{n * Q * R^{2/3}}{A}\right]^2 \tag{4}$$

can be used. In these relations, Q is the discharge, A is the area of cross section, B is the width of water surface, g is the acceleration of gravitation, R is the hydraulic radius, C and n are the Chezy and Manning coefficients.

Statements of flow area (A) which is defined as dependent on depth and wetted perimeter (U) change along the channel in uniform, prismatic channels.

Expanding Equation (1) for trapezoidal cross sections gives the following equation:

$$\frac{dy}{dx} = \frac{So - \left[\frac{n * Q * (\mathrm{b} + 2\mathrm{y}\sqrt{1+\mathrm{m}^2})^{2/3}}{(b+my)^{5/3} y^{5/3}}\right]^2}{1 - \frac{Q^2 (b+2my)}{g(b+my)^3 y^3}} \tag{5}$$

Looking at the change in the water surface profile, different profiles are confronted in the channel depending on the slope of channel bottom and boundary condition. Since it is known that, at which intervals the depth changes in these profiles, x values can be calculated by selecting *y* values in Equation (5).

In non-uniform channels, it is possible that there may be different cross sections at different points of the channel also there may be local energy losses that must be taken into consideration. In this case, instead of calculating the distance based on the depth, it is necessary that depths be calculated at desired intervals (cross sections).

In Direct Step Method (DSM), the following energy equation is written between the two cross sections

$$\frac{V_1^2}{2g} + y_1 + z_1 = \frac{V_2^2}{2g} + y_2 + z_2 + \frac{\Delta x}{2}\left(Sf_1 + Sf_2\right) \tag{6}$$

and the y_2 value that will satisfy the equation is obtained through iteration (Chow, 1955; Chaudry, 1993).

Differential quadrature method (DQM)

Differential Quadrature Method (DQM), first proposed by Bellman et al. (1971), is an alternative approach to the standard numerical solution methods such as finite difference and finite elements, for the initial and boundary value problems encountered in physics and mathematics (Shu and Richards, 1992; Shu et al., 2003; Shu and Chew, 1997). In DQM, the partial derivative of a function with respect to a variable at a discrete point is approximated as a weighted linear sum of the function values at all discrete points in the region of that variable. The approximation of the partial derivative can be written as:

$$u_x^{(r)}(x_i) = \left.\frac{\partial^r u}{\partial x^r}\right|_{x=xi} = \sum_{j=1}^{N} A_{ij}^{(r)} u(x_j) \qquad i = 1,2,\ldots,N \tag{7}$$

where $u_x^{(r)}$ is the rth order derivative of the function, x_j are the discrete points of the variable x, $u(x_j)$ are the function values at points x_j and $A^{(r)}_{ij}$ are the weight coefficients for the rth order derivative of the function (Civalek, 2004).

Determining the weight coefficients is the most crucial step in use of DQM. Shu and Xue (1997) worked on the selection of the weight coefficients and proposed several solutions in their studies. The weight coefficients change upon approximation function and according to the chosen approximation function the method takes different names such as Polynomial Differential Quadrature (PDQ), Fourier Expansion Base Differential Quadrature (FDQ) and Harmonic Differential Quadrature (HDQ) (Civalek, 2004; Shu et al, 2002).

For boundary value problems, DQM performance is highly dependent on the boundary conditions and sampling grid points. The boundary conditions can be easily implemented to DQ system and the common type of boundary conditions, which are Dirichlet, Neumann and/or mixed type function, do not create any difficulty in this implementation process (Bert and Malik, 1996; Civalek 2003).

The overall sensitivity of the model especially depends on the location and number of sampling grid points. However, Civalek (2003) points out that the determination of the effective choice of sampling grid points for any problem reduces the analysis time. For instance, the previous studies have showed that for the solution of linear equations with homogeneous boundary conditions, selecting equal intervals between the adjacent grid points are adequate. On the other hand, for the vibration problems, the choice of grid points through the Chebyshev-Gauss-Lobatto method is more reasonable. In time-bound equations and initial value problems, selection of unequal intervals for sampling grid points produces the appropriate solutions.

Solution of differential equation of water surface through DQM

Uniform channels

In solutions through DQM, it is necessary that calculation points are known beforehand. Looking at the change of the water surface profile, boundary values of different type of profiles are known. For this reason, calculation for distances of x has been realized by choosing values of depth. Since y values are determined in the choice of calculation points, it is necessary to write the Equation (5) as follows

$$\frac{dx}{dy}=\frac{1-\dfrac{Q^2(b+2my)}{g(b+my)^3y^3}}{So-\left[\dfrac{n*Q*(\mathrm{b}+2\mathrm{y}\sqrt{1+\mathrm{m}^2})^{2/3}}{(b+my)^{5/3}y^{5/3}}\right]^2} \qquad (8)$$

C_{ij} is the matrix of weighed point, and writing it as follows:

$$\sum_{j=1}^{N}C_{ij}x_j=\left.\frac{1-\dfrac{Q^2B}{gA^3}}{So-\left[\dfrac{n*Q*U^{2/3}}{A^{5/3}}\right]^2}\right|_i \qquad i=1,2,3,...N \qquad (9)$$

by solving the set of equations, values of x are calculated corresponding to the chosen values of y. distribution proposed by Shu and Chew (2003) was applied in choosing the depth of calculation. In this distribution, l_i: shows the position of the point i in the calculation interval

$$r_i=\frac{1}{2}\left(1-\cos\left(\frac{i-1}{N-1}\pi\right)\right) \qquad i=1,2,3,...N \qquad (10a)$$

$$l_i=\left((1-\alpha)(3r_i^2-2r_i^3)+\alpha r_i^2\right)\alpha \qquad (10b)$$

α is the coefficient which controls the distribution of points α<1, on the other hand, means that points get dense on boundaries, and α>1 means that points become sparse at boundaries.

Non-uniform channels

In non-uniform channels, depth must be calculated at desired intervals (cross sections). For this reason, when choosing points of calculation by DQM, it is necessary to choose intervals of x then to calculate intervals of y.

Taking the width of water surface B, wetted perimeter U, wetted area A as the values at cross section, values at the i cross section, C_{ij} weighted coefficient matrix can be written as follows;

$$\left.\frac{dy}{dx}\right|_i=\sum_{j=1}^{N}C_{ij}y_j=\left.\frac{So-\left[\dfrac{n*Q*U^{2/3}}{A^{5/3}}\right]^2}{1-\dfrac{Q^2B}{gA^3}}\right|_i \qquad i=1,2,3,...N \qquad (11)$$

Solving the set of equations obtained, values of y are calculated against the values of x chosen. Since we face a non-linear set of equations here, values of y can be determined by solving this equation by using Newton-Raphson method.

For a cross section at which local energy loss is at stake, by defining two points for one cross section and using energy equation, one of the points can be defined at the kind of the other and added into the set of equations.

Defining the boundary condition is realized by putting the known values of depth y into the equation and taking out from the set of equations the equation involving y value of the related cross section.

Local head loses

In a cross section having head loses at a channel, the local head lose is defined in this section, and the test function is written both before and after the section.

At a problem with N calculation points, if any point, such as section M, has a local head lose, Equation (11) can be written as Equation (12):

$$\left.\frac{dy}{dx}\right|_i=\sum_{j=1}^{M}C_{ij}y_j=\left.\frac{So-\left[\dfrac{n*Q*U^{2/3}}{A^{5/3}}\right]^2}{1-\dfrac{Q^2B}{gA^3}}\right|_i \qquad i=1,2,3,...M$$

$$y_M=y_{M+1}+K\frac{Q^2}{2gA^2}$$

$$\left.\frac{dy}{dx}\right|_i=\sum_{j=M+1}^{N+1}C_{ij}y_j=\left.\frac{So-\left[\dfrac{n*Q*U^{2/3}}{A^{5/3}}\right]^2}{1-\dfrac{Q^2B}{gA^3}}\right|_i \qquad i=M+1,M+2,M+3,...N+1$$

(12)

In this equation, K is the local head lose coefficient. The water depth can be calculated by the solution of the system of equations.

Numerical solution

A uniform prismatic channel was chosen for numerical solution. A trapezoidal channel was selected for examples in which two different channel bottom slopes are taken, one being low and the other being high. Bottom width is 5 m, bank slopes being 1:m=1:1, Manning roughness coefficient being 0.02, and discharge being 20 m^3/s. Channel bottom slopes were taken as 0.01 and 0.001. Computations are carried out for 20 points.

Since computation points are different on DQM and DSM, elevation values are determined by using interpolation in order to

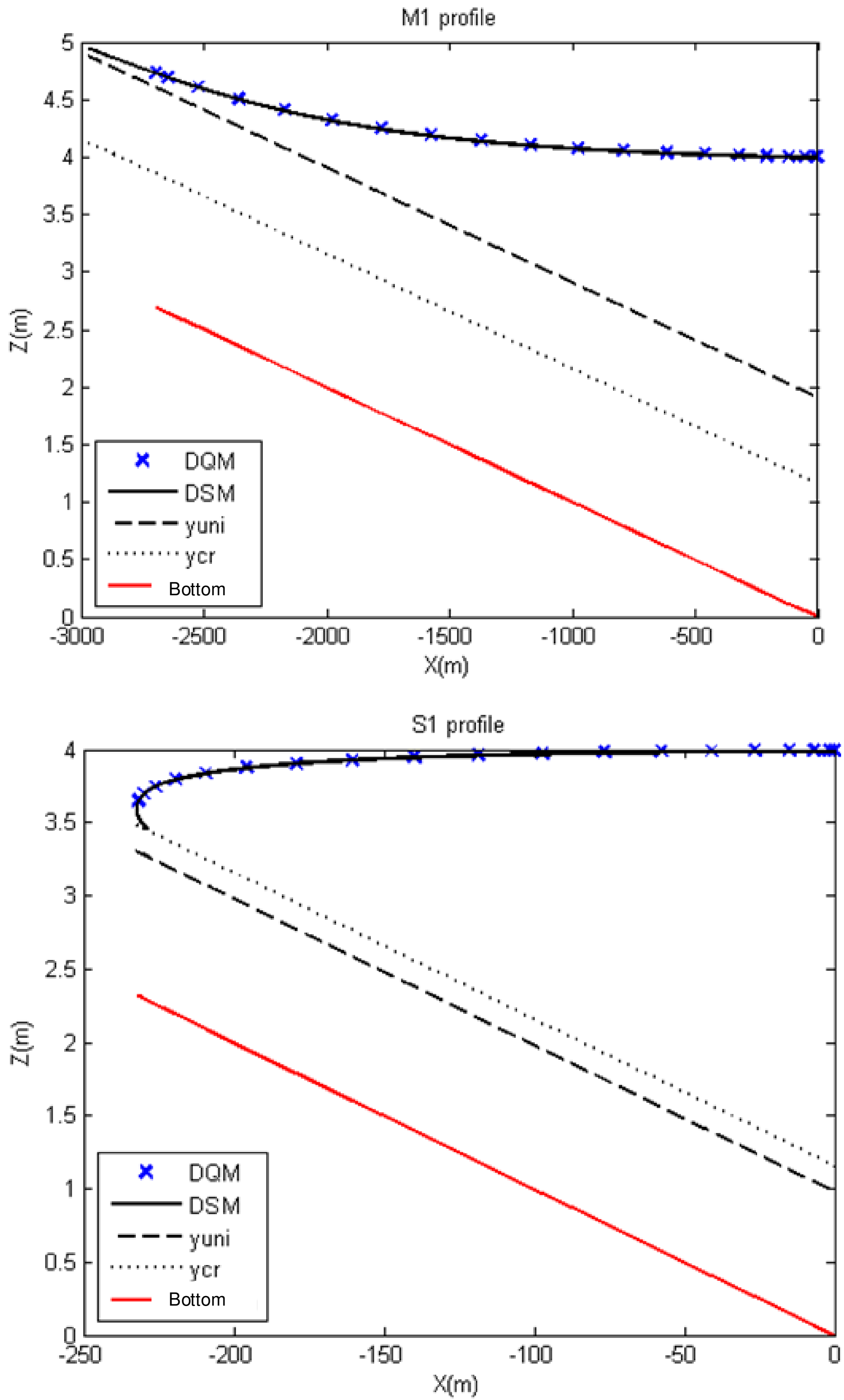

Figure 1. Water surface profiles.

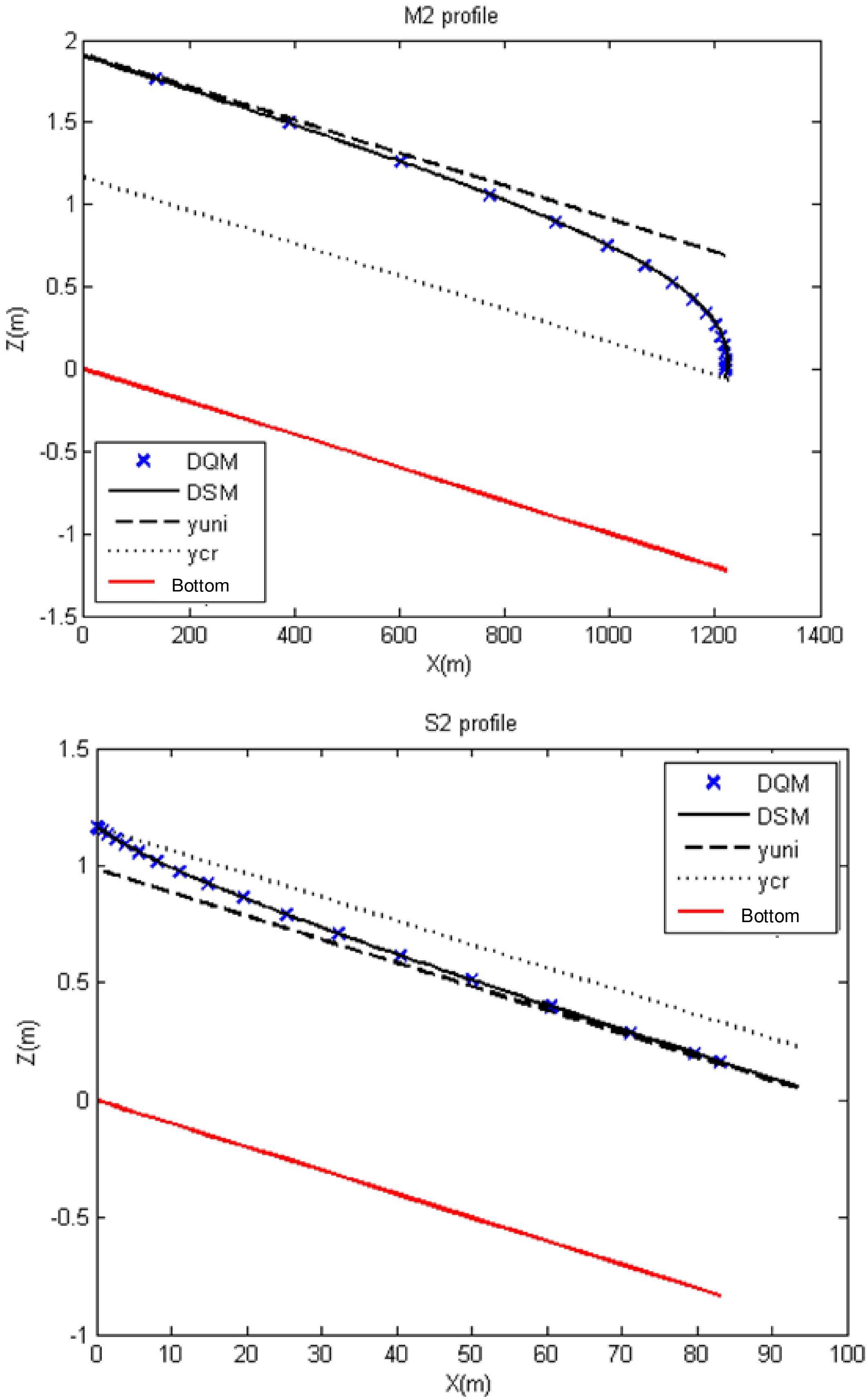

Figure 1. Contd.

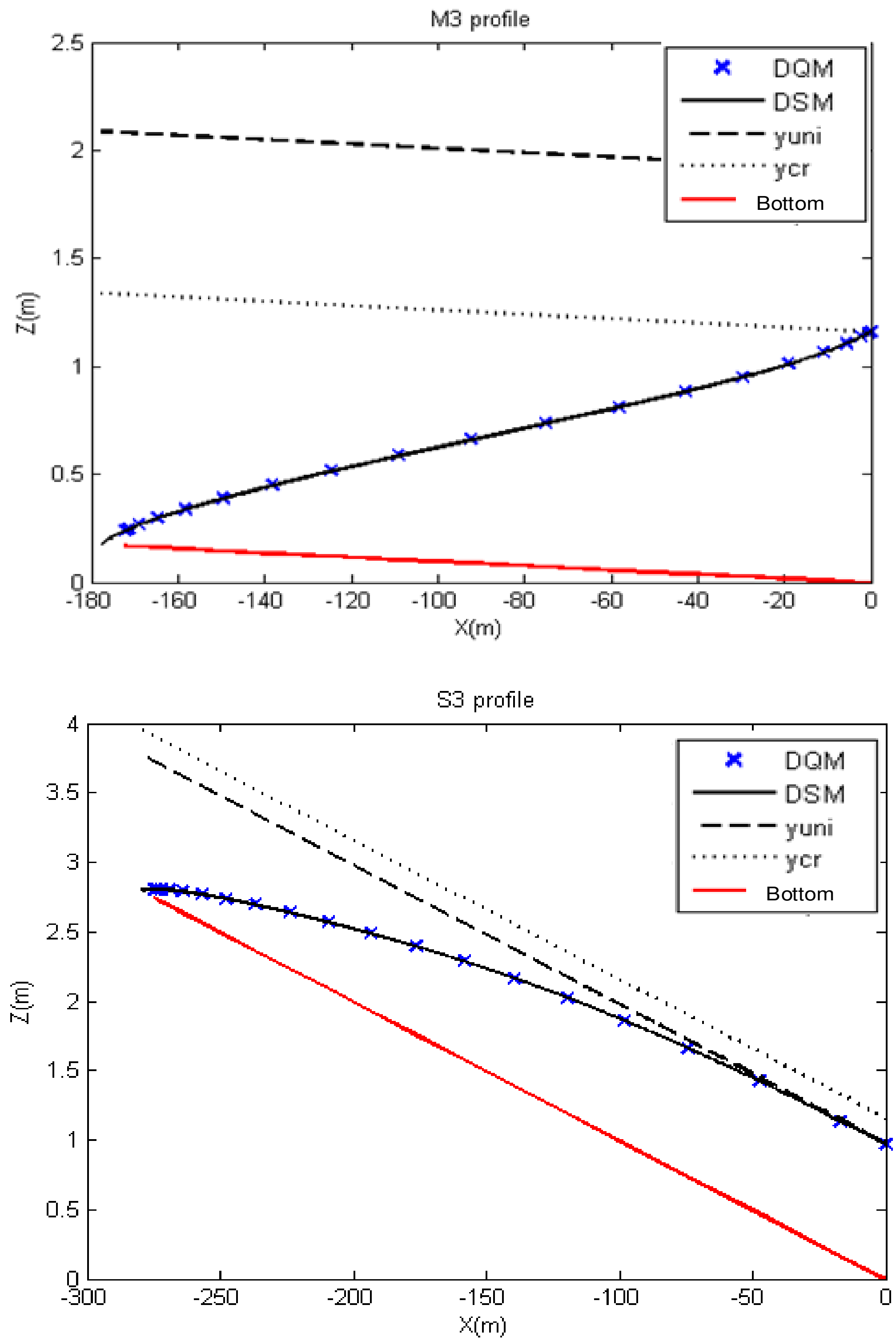

Figure 1. Contd.

compare the results. Results obtained through both DSM and DQM are shown in Figure 1 for the numerical example. Also, differences between results of DSM and DQM methods are presented in Tables 1 and 2.

RESULTS

Differences between water surface elevations obtained

Table 1. Differences between water surface elevations obtained by using the DSM and DQM methods for M type profile.

M1 profile			M2 profile			M3 profile		
X (m)	Differences(cm)	Relative differences (%)	X(m)	Differences(cm)	Relative differences (%)	X (m)	Differences (cm)	Relative differences (%)
0.0	0.00	0.0E+00	-2065.3	0.00	0.0E+00	176.4	0.00	0.0E+00
-16.8	-0.23	1.4E-01	-1556.6	0.07	4.0E-02	175.8	-1.46	8.5E-01
-57.2	0.00	1.5E-03	-944.4	-0.09	5.2E-02	173.8	-1.30	7.6E-01
-130.2	-0.24	1.4E-01	-670.6	0.11	6.5E-02	170.2	-1.11	6.5E-01
-224.4	0.01	5.5E-03	-458.2	-0.09	5.0E-02	164.3	-0.95	5.6E-01
-348.3	-0.24	1.4E-01	-325.2	0.15	9.0E-02	155.7	-0.83	4.9E-01
-488.9	0.02	1.0E-02	-217.9	-0.06	3.7E-02	144.3	-0.75	4.4E-01
-654.4	-0.24	1.4E-01	-147.2	0.22	1.3E-01	130.4	-0.69	4.1E-01
-830.2	0.02	1.3E-02	-91.1	0.00	2.6E-03	114.4	-0.66	3.9E-01
-1024.9	-0.23	1.4E-01	-55.4	0.35	2.1E-01	97.1	-0.65	3.8E-01
-1223.3	0.02	1.2E-02	-28.7	0.14	8.4E-02	79.5	-0.65	3.8E-01
-1434.7	-0.21	1.2E-01	-13.3	0.66	3.8E-01	62.4	-0.67	3.9E-01
-1645.4	0.01	7.4E-03	-3.0	0.60	3.5E-01	46.6	-0.71	4.1E-01
-1866.1	-0.16	9.6E-02	1.4	1.73	1.0E+00	32.9	-0.76	4.4E-01
-2086.7	0.00	6.9E-04	3.5	3.15	1.8E+00	21.5	-0.83	4.9E-01
-2320.3	-0.10	6.0E-02	3.3	-3.61	2.1E+00	12.7	-0.94	5.5E-01
-2561.6	-0.01	4.4E-03	2.4	-2.12	1.2E+00	6.5	-1.09	6.4E-01
-2820.9	-0.04	2.4E-02	1.2	-1.73	1.0E+00	2.6	-1.33	7.8E-01
-3067.2	-0.01	6.6E-03	0.3	-1.51	8.9E-01	0.6	-1.71	1.0E+00
-3189.4	-0.01	6.4E-03	0.0	-1.46	8.6E-01	0.0	-2.32	1.4E+00

by using the DSM and DQM methods are calculated and relative differences (Equation13) are determined as shown below.

$$RelativeDifference(\%) = \frac{\left| y_{DQM} - y_{DSM} \right|}{y_{uniform}} 100 \quad (13)$$

Relative differences are given in Tables 1 and 2.

In an illustrative example given in this study, the differences between DQM and DSM results are very small. In the M1, M2 and M3 profiles, maximum relative differences are 0.14, 2.1 and 1.4%, respectively. Similarly, the maximum relative differences are 1.8E-4, 0.32 and 2.1% in the S1, S2 and S3 profiles. At DQM applications, similar results can be obtained for non-uniform channels with or without local head loses.

DISCUSSION

Depending on the number of calculation points, there may appear very small differences between the results obtained through DQM and those obtained through DSM. The differences between both methods converge depending on the number of points. In DSM, the depth value at any cross section is calculated based on the depth values in the preceding cross section. In fact, a depth in a cross section is not only based on the depth at the preceding point but also the following point. Moreover, the depth in any cross section in DQM is expressed as a function of all other cross sections.

CONCLUSION

The application of the method is very easy in uniform channels. In the problem having N calculation

Table 2. Differences between water surface elevations obtained by using the DSM and DQM methods for S type profile.

S1 profile			S2 profile			S3 profile		
X (m)	Differences (cm)	Relative differences (%)	X (m)	Differences (cm)	Relative differences (%)	X (m)	Differences (cm)	Relative differences(%)
0.0	0.00	0.0E+00	0.0	0.00	0.0E+00	0.0	0.00	0.0E+00
-1.2	0.00	5.8E-05	0.3	-0.28	3.2E-01	-49.1	0.07	8.1E-02
-4.6	0.00	8.8E-07	0.4	0.00	3.6E-03	-110.2	-0.17	2.0E-01
-10.2	0.00	5.9E-05	1.1	-0.25	2.8E-01	-141.7	-0.02	2.3E-02
-17.9	0.00	2.4E-06	1.7	0.01	1.0E-02	-169.4	-0.28	3.2E-01
-27.4	0.00	6.1E-05	3.0	-0.20	2.3E-01	-191.6	-0.18	2.0E-01
-38.4	0.00	4.7E-06	4.2	0.01	1.5E-02	-213.0	-0.39	4.4E-01
-50.5	0.00	6.3E-05	6.4	-0.15	1.7E-01	-232.2	-0.32	3.6E-01
-63.5	0.00	4.6E-06	8.6	0.01	1.4E-02	-250.9	-0.46	5.2E-01
-76.7	0.00	6.8E-05	12.2	-0.10	1.2E-01	-268.1	-0.42	4.8E-01
-89.7	0.00	4.6E-06	16.0	0.01	8.6E-03	-284.4	-0.52	5.9E-01
-101.9	0.00	6.9E-05	21.4	-0.06	7.0E-02	-299.2	-0.51	5.8E-01
-112.7	0.00	3.6E-06	27.6	0.00	3.7E-03	-312.4	-0.59	6.7E-01
-121.8	0.00	7.9E-05	35.9	-0.03	3.7E-02	-323.6	-0.61	7.0E-01
-128.6	0.00	1.9E-05	45.7	0.00	6.0E-04	-332.9	-0.70	7.9E-01
-132.9	0.00	1.2E-04	58.4	-0.01	1.5E-02	-339.9	-0.78	8.8E-01
-134.8	0.00	1.8E-04	73.8	0.00	8.2E-04	-344.9	-0.91	1.0E+00
-134.7	0.00	1.4E-04	92.9	0.00	4.7E-03	-348.0	-1.11	1.3E+00
-133.9	0.00	5.0E-05	112.9	0.00	6.5E-04	-349.6	-1.43	1.6E+00
-133.4	0.00	7.0E-05	123.3	0.00	1.5E-03	-350.1	-1.88	2.1E+00

points, the water depths can be determined over the solution of the linear equations system. However, in non-uniform channels, non-linear system of equations is required. In this case, DQM needs iteration also like other numerical methods. However, the advantage of DQM is using less grid points than other numerical methods. In addition, more accurate results are obtained. Depending on a problem, the number of calculation points (N) required is changed. In the present study, the results of DQM with 15 grid points are almost equal with DSM's with 90 points.

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