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Review

Life-time portfolio selection model

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In the field of financial economics, many researchers have developed optimal portfolio selection models. Unfortunately, those models cannot be implemented directly by investors since the coefficient of risk aversion is an exogenous variable. However, even if investors have no idea about their attitudes toward risk, investors might specify a maximum probability of failing to reach a specific portfolio threshold. According to this argument, we regard stop-loss level as portfolio threshold and penetrate the attitude towards risk through the stop-loss level. Then, we can achieve investor's optimal life-time portfolio selection model.

Key words: Dynamic asset allocation, portfolio optimization, value-at-risk, stop-loss.

INTRODUCTION

Merton (1969, 1971) was the pioneer of using continuous-time modeling in financial economics by formulating the intertemporal consumption and portfolio choice problem of an investor in a stochastic dynamic programming setting. After the development of simple relation between consumption and asset returns by Lucas (1978) and Breeden (1979), consumption-based asset pricing theory became one of the major advances in financial economics over the past decades. Many researchers Bodie et al. (1992), Hindy et al. (1993), Kim and Omberg (1996), Brennan et al. (1997), Sorensen (1999), Lioui and Poncet (2001), Viceira (2001), Xia (2001), Wachter (2002), Yen and HsuKu (2003), Bajeux-Besnainou et al. (2003), Chacko and Viceira (2005), Guo and Yen (2006,2008), Guo (2009), Benzoni et al. (2007), Garlappi et al. (2007), Kan and Zhou (2007), Liu (2007), and HsuKu (2007) have extended Merton's dynamic asset allocation model over the past decades. Unfortunately, in those models, the risk aversion coefficient is an exogenous variable. Hence, those models cannot be implemented directly by investors. Given specified stop-loss level and value-at-risk, we can find out the risk aversion coefficient through investor's specified threshold and probability of failing to reach the portfolio threshold.

The St. Petersburg Paradox offers some clues about investors' attitude toward risk. The game is played by flipping a coin until it comes up tails, and the total number of flips, *n*, determines the prize, which equals $\$2^n$. Thus, if the coin comes up tails the first time, the prize is $\$2^1 = \$2$, and the game ends. If the coin comes up heads the first time, it is flipped again. If it comes up tails the second time, the prize is $\$2^2 = \$4$, and the game ends. If it comes up heads the second time, it is flipped again, and so on. The game consists of an infinite number of possible consequences (runs of heads followed by one tail). Since the expected payoff of each possible consequence is \$1, and there are an infinite number of them, the expected value of the game is an infinite number of dollars. The classical resolution of the paradox involves an explicit introduction of a utility function. Although, investors typically cannot make sure of their attitudes toward risk, they can always evaluate the value of the game. If the utility function of investors is given by $U(W) = -W^{-\gamma},\ \gamma > 0$, the expected utility of the St. Petersburg game is:

$$E[U(W)] = -\frac{1}{2^1}\times 2^{-\gamma} - \frac{1}{2^2}\times 2^{-2\gamma} - \frac{1}{2^3}\times 2^{-3\gamma} - \frac{1}{2^4}\times 2^{-4\gamma} - \cdots = \frac{-2^{-(\gamma+1)}}{1-2^{-(\gamma+1)}}$$

We can indirectly uncover γ by the evaluated value. For example, the evaluation of \$ 3 indicates γ equals 1 ($-2^{-(\gamma+1)}/\left(1-2^{-(\gamma+1)}\right) = -3^{-\gamma} \Rightarrow \gamma = 1$). Hence, investor's attitude toward risk can be uncovered by specified stop-loss level and value-at-risk. Once the risk tolerance is discovered, we can determine investor's optimal life-time portfolio selection model.

THE ECONOMIC SETTING

In the market, N risky assets and one risk-free asset are assumed and all of these securities may be infinitely divided with the returns accrued only in the form of capital gains (no dividend payout). Taxes, transaction costs, and short-sell constraints are all inapplicable.

Assuming the price of the jth asset at time t, S_{jt}, follows the Ito process with the following differential equation:

$$\frac{dS_{jt}}{S_{jt}} = \mu_j dt + \sigma_j dz_j \tag{1}$$

Where z_j is a Wiener process; μ_j is the expected instantaneous rate of return of the jth risky asset at time t; σ_j is the standard deviation of expected instantaneous rate of return of the jth risky asset at time t.

Let B_t be the total amount of the risk-free asset that the investor holds at time t and the dynamics for B_t is given by:

$$\frac{dB_t}{B_t} = r_f dt \tag{2}$$

where r_f is the expected instantaneous rate of return of the risk-free asset.

Let W_t be the total wealth held by an investor at time t, comprising the formula:

$$W_t = \sum_{j=1}^{N} n_{jt} S_{jt} + B_t \tag{3}$$

where n_{jt} is the number of shares of the jth risky asset held by the investor at time t.

By Equations 1, 2, and 3, we have the dynamic stochastic process of the total wealth:

$$dW_t = \sum_{j=1}^{N} w_{jt} \left[\left(\mu_j - r_f \right) dt + \sigma_j dz_j \right] W_t + r_f W_t dt - C_t dt \tag{4}$$

where C_t is the consumption of the investor at time t; w_{jt} is the proportion of the total wealth that the investor invests in the jth risky asset at time t, $j = 1, \ldots, N$.

The first moment and second moment of Equation 4 are as:

$$\begin{aligned} E_t(dW_t) &= \sum_{j=1}^{N} w_{jt} (\mu_j - r) W_t dt + rW_t dt - C_t dt \\ &= \mathbf{w_t}' \boldsymbol{\mu}_\mathbf{t} W_t dt + rW_t dt - C_t dt \end{aligned} \tag{5}$$

$$E_t\left((dW_t)^2 \right) = V_t(dW_t) + \left(E_t(dW_t) \right)^2 = \mathbf{w_t}' \boldsymbol{\Sigma}_\mathbf{t} \mathbf{w_t} W_t^2 dt \tag{6}$$

where $\mathbf{w_t}$ is the $N \times 1$ vector with representative elements w_{jt}; $\mathbf{w_t}'$ is the transpose of $\mathbf{w_t}$; $\boldsymbol{\mu}_\mathbf{t}$ is the $N \times 1$ vector of expected instantaneous rate of excess return of risky assets at time t; $\boldsymbol{\Sigma}_\mathbf{t}$ is the $N \times N$ variance-covariance matrix of the expected instantaneous rate of return of risky assets at time t.

THE MODEL

To determine an individual investor's optimal asset allocation strategy in a stochastic dynamic programming setting, we must apply a utility function in advance. In this paper, we adopt the power utility function and take the utility function of investors as:

$$U(C_t) = -C_t^{-\beta} = -\left[C(W_t) \right]^{-\beta}, \quad \beta > 0 \tag{7}$$

This is a well-known strictly concave power utility function, that is, $U'(C_t) > 0$ and $U''(C_t) < 0$, and $1+\beta$ is the coefficient of relative risk aversion.

Assume that the investor desires to solve the following dynamic portfolio choice problem:

$$\underset{C_t, \mathbf{w_t}'}{Max} \; E_t \left[\int_t^T e^{-\delta \tau} U(C_\tau) \, d\tau + B(W_T, T) \right] \tag{8}$$

substituting Equations 4, 5, 6 and 7:

$C_t > 0$, $W_t > 0$

where $B(W_T, T)$ is the bequest function. Let $J = J(W_t, t)$ be the well-behaved function such that:

$$J = \underset{C_t, \mathbf{w_t}'}{Max} E_t \left[\int_t^T e^{-\delta \tau} U(C_\tau) d\tau + B(W_T, T) \right]$$

The Hamilton-Jacobi-Bellman (HJB) equation is:

$$0 = \underset{C_t, \mathbf{w_t}'}{Max} \left\{ e^{-\delta t} U(C_t) + J_W \left(\mathbf{w_t}' \boldsymbol{\mu}_\mathbf{t} W_t + r_f W_t - C_t \right) + J_t + \frac{1}{2} J_{WW} \mathbf{w_t}' \boldsymbol{\Sigma}_\mathbf{t} \mathbf{w_t} W_t^2 \right\} \tag{9}$$

where J_W denotes the derivative of J with respect to W_t , with a similar notation used for higher derivatives; J_t denotes the derivative of J with respect to t.

The first order conditions to Equation 9 are:

$$\begin{cases} U_{C_t} = J_W \\ \mathbf{w_t} = -\dfrac{J_W}{J_{WW} W_t} \boldsymbol{\Sigma}_\mathbf{t}^{-1} \boldsymbol{\mu}_\mathbf{t} \end{cases}$$

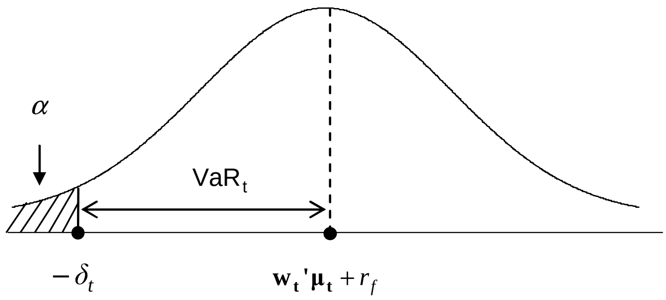

Figure 1. The relationship between stop-loss level and VaR.

Then, we can yield the optimal dynamic asset allocation strategy for the investor as:

$$\mathbf{w_t} = \frac{1}{1+\beta}\mathbf{\Sigma_t}^{-1}\mathbf{\mu_t} \quad (10)$$

SHARPE RATIO

Sharpe (1966, 1975, 1994) introduced a measure for the performance of mutual funds and proposed the term reward-to-variability ratio to measure risk-adjusted performance. The Sharpe ratio is calculated by subtracting the risk free rate from the return of the portfolio and then dividing by the portfolio's standard deviation. In Equation 10, the rate of excess return and variance are $\mathbf{w_t}'\mathbf{\mu_t}$ and $\mathbf{w_t}'\mathbf{\Sigma_t}\mathbf{w_t}$, respectively. Let SR_{pt} be the Sharpe ratio of our model at time *t*, we have:

$$SR_{pt} = \frac{\mathbf{w_t}'\mathbf{\mu_t}}{(\mathbf{w_t}'\mathbf{\Sigma_t}\mathbf{w_t})^{1/2}} = \left(\mathbf{\mu_t}'\mathbf{\Sigma_t}^{-1}\mathbf{\mu_t}\right)^{\frac{1}{2}} \quad (11)$$

It is obvious that the Sharpe ratio of our model is uncorrelated with investor's attitude towards risk.

RISK AVERSION COEFFICIENT

Since the coefficient of relative risk aversion has puzzled investors, we introduce the stop-loss level and the maximum probability of failing to reach the threshold to solve the risk aversion coefficient. In Shefrin and Statman (2000) and Das et al. (2010), investors maximize expected returns subject to a constraint that the probability of failing to reach a threshold level not exceeds a specified maximum probability. It is the same as expected wealth optimization with a value-at-risk constraint.

Value-at-risk (VaR, hereafter) has emerged as the standard tool for measuring and managing financial market risk. The concept of VaR as a single risk measure summarizing all sources of downside risk was first developed by Morgan and made available through its Risk-Metrics software in October 1994. For a given portfolio, probability and time horizon, VaR is defined as a threshold value such that the probability that the loss on the portfolio over the given time horizon exceeds this value is the given probability. For example, if a portfolio has a daily 95% VaR of $1 million with 95% confidence level, it means that over the next 24-h period, there is a 5% probability that the portfolio will fall in value by more than $ 1 million. By Equation 1, we have:

$$d\ln S_{jt} \sim N\left(\mu_j dt, \sigma_j^2 dt\right) \quad (12)$$

The existing literature suggests capturing the uncertainty in VaR estimates in the form of VaR confidence intervals. Jorion (1996) suggests normal distributed returns. Hence, the instantaneous rate of excess return of investor's optimal life-time portfolio follows a normal distribution. Its mean and variance are $\mathbf{w_t}'\mathbf{\mu_t}$ and $\mathbf{w_t}'\mathbf{\Sigma_t}\mathbf{w_t}$, respectively. Therefore, at time *t*, VaR of the investor's optimal life-time portfolio is:

$$\text{VaR}_t = z_\alpha(\mathbf{w_t}'\mathbf{\Sigma_t}\mathbf{w_t})^{1/2} W_t \quad (13)$$

where z_α is the standard normal variable with cumulative probability of α.

Stop-loss strategies are widely used trading strategies among financial practitioners. It can prevent investors from holding their losing investments too long by automatically prompting the sales of losing investments. Brown et al. (2010) show that trailing stop-loss orders produce positive abnormal returns with respect to the Standard and Poor's 500 market index as a benchmark and protect investors as a mean-variance efficient strategy. The simplest stop-loss strategy involves setting a sell level at a fixed percentage below the purchase price at the time of entry.

Let δ_t be the tolerable loss ratio of the investor at time *t*. If the investor specify the return on the portfolio should not fall below $-\delta_t$ with more than α probability. As shown in Figure 1, we have:

$$\delta_t + \mathbf{w_t}'\mathbf{\mu_t} + r_f = z_\alpha(\mathbf{w_t}'\mathbf{\Sigma_t}\mathbf{w_t})^{1/2} \quad (14)$$

By Equations 10, 11 and 14, the coefficient of relative risk aversion is:

$$1+\beta = \frac{SR_{pt}}{\delta_t + r_f}\left(z_\alpha - SR_{pt}\right) \quad (15)$$

OPTIMAL LIFE-TIME PORTFOLIO SELECTION MODEL

After substituting Equation 15 into Equation 10, we have the optimal portfolio selection model as:

$$\mathbf{w_t} = \frac{\delta_t + r_f}{SR_{pt}\left(z_\alpha - SR_{pt}\right)}\mathbf{\Sigma_t}^{-1}\boldsymbol{\mu}_\mathbf{t} \quad (16)$$

Equation 16 shows that the optimal tailor-made life-time portfolio selection model is composed of the tolerable loss ratio, the probability of failing to reach the tolerable loss ratio, the rate of expected return of the risk-free asset, and the rate of expected excess return and risk of risky securities. Moreover, all of those elements can be determined before you make an investment decision.

Conclusion

Over the past decades, many researchers have developed optimal portfolio selection models. However, those models cannot be implemented directly by investors since the attitude toward risk has puzzled investors. In this paper, we demonstrate that all of the elements in our model can be determined before making an investment decision. Therefore, investors can make good use of our optimal life-time portfolio selection model.

REFERENCES

Bajeux-Besnainou I, Jordan JV, Portait R (2003). Dynamic asset allocation for stocks, bonds, and cash. J. Bus., 76: 263-287.

Benzoni LP, Collin-Dufresne, Goldstein RS (2007). Portfolio choice over the life-cycle when the stock and labor markets are cointegrated. J. Financ., 62(5): 2123-2167.

Bodie Z, Merton RC, Samuelson WF (1992). Labor supply flexibility and portfolio choice in a life cycle model. J. Econ. Dynam. Control, 16(3-4): 427-449.

Breeden DT (1979. An intertemporal asset pricing model with stochastic consumption and investment opportunities, J. Finan. Econ., 7: 265-296.

Brennan MJ, Schwartz ES, Lagnado R (1997). Strategic asset allocation. J. Econ. Dynam. Control, 21: 1377-1403.

Brown S, Cao Alvira JJ, Pérez J (2010). Do stop-loss orders protect Investors? J. Acad. Bus. Econ., 10(1): 65-77

Chacko G, Viceira LM (2005). Dynamic Consumption and Portfolio Choice with Stochastic Volatility in Incomplete Markets. Rev. Finan. Stud., 18: 1369-1402.

Das S, Markowitz H, Scheid J, Statman M (2010). Portfolio optimization with mental accounts. J. Finan. Quant. Anal., 45(2): 311-334.

Garlappi L, Uppal R, Wang T (2007). Portfolio selection with parameter and model uncertainty: A multi-prior approach. Rev. Finan. Stud., 20(1): 41-81.

Guo Z, Yen SH (2006). Determining Institutional Investor' dynamic asset allocation. J. Finan. Stud., 14: 77-94.

Guo Z, Yen SH (2008). Portfolio Selection of Institutional Investor Based on Value Function. J. Manage., 25(1): 31-49.

Guo Z, (2009). International Portfolio Holdings of Institutional Investors. Asia J. Manage. Humanity Sci., 4(1): 16-27.

Hindy A, Huang CF (1993). Optimal consumption and portfolio rules with durability and local substitution. Econometrica., 61(1): 85-121.

Hsuku YH (2007). Dynamic Consumption and Asset Allocation with Derivative Securities. Quant. Finan., 7: 137-149.

Jorion P (1996). Risk2: Measuring the risk in value at risk. Finan. Anal. J., 52: 47-56.

Kan R, Zhou G (2007). Optimal portfolio choice with parameter uncertainty. J. Finan. Quant. Anal. 42(3): 621-656.

Kim T, Omberg E (1996). Dynamic non-myopic portfolio behavior. Rev. Finan. Stud., 9: 141-161.

Lioui A, Poncet P (2001). On optimal portfolio choice under stochastic interest rates. J. Econ. Dynam. Control, 25: 1841-1865.

Liu J (2007). Portfolio Selection in Stochastic Environments. Rev. Finan. Stud., 20: 1-39.

Lucas RE Jr (1978). Asset prices in an exchange economy, Econometrica., 46: 1429-1445.

Merton RC (1969). Lifetime portfolio selection under uncertainty: the continuous case. Rev. Econ. Statist., 51(3): 247-257.

Merton RC (1971). Optimum consumption and portfolio rules in a continuous-time model. J. Econ. Theory, 3: 373-413.

Sharpe WF (1966). Mutual Fund Performance. J. Bus., 39: 572-592.

Sharpe WF (1975). Adjusting for Risk in Portfolio Performance Measurement. J. Portfol. Manage., Winter: 29-34.

Sharpe WF (1994). The Sharpe Ratio. J. Portfol. Manage., Fall: 49–58.

Sorensen C (1999). Dynamic asset allocation and fixed income management. J. Financ. Quant. Anal., 34: 513-532.

Shefrin H, Statman M (2000). Behavioral portfolio theory. J. Financ. Quant. Anal., 35(2): 127-151.

Viceira LM (2001). Optimal portfolio choice for long-horizon investors with nontradable labor income. J. Finan., 56: 433-470.

Wachter JA (2002). Portfolio and Consumption Decisions under Mean-Reverting Returns: An Exact Solution for Complete Markets. J. Financ. Quant. Anal., 37: 63-91.

Xia Y (2001). Learning about predictability: The effects of parameter uncertainty on dynamic asset allocation. J. Financ., 56: 205-246.

Yen SH, HsuKu YH (2003). Dynamic asset allocation strategy for intertemporal pension fund management with time-varying volatility. Academic Econ. Pap., 31: 229-261.