Subclasses of analytic functions associated with Wright generalized hypergeometric functions

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In this paper, we define a generalized class of starlike functions with negative coefficients and obtain coefficient estimates, distortion bounds, closure theorems and extreme points. Further we obtain modified Hadamard product, radii of close-to-convex, starlikeness and convexity for functions belonging to this class. Furthermore neighborhood results are discussed.

Key words: Univalent functions, convex functions, Starlike functions, $\delta$-neighbourhood, inclusion relations, Hadamard product, Wright generalized hypergeometric functions.

INTRODUCTION AND PRELIMINARIES

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic and univalent in the open disc $U = \{z: |z| < 1\}$ and normalized by $f(0) = 0 = f'(0) - 1$. We denote by $S^+(\alpha)$ and $K(\alpha)$ the subclasses of $A$ consisting of all functions which are, respectively starlike and convex of order $\alpha$. Thus,

$$S^+(\alpha) = \left\{ f \in A : \text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, 0 \leq \alpha < 1, \ z \in U \right\}$$

And

$$K(\alpha) = \left\{ f \in A : \text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, 0 \leq \alpha < 1, \ z \in U \right\}$$

For functions $\Phi \in A$ given by $\Phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$ and $\Psi \in A$ given $\Psi(z) = z + \sum_{n=2}^{\infty} \psi_n z^n$ we define the Hadamard product (or convolution) of $\Phi$ and $\Psi$ by

$$(\Phi * \Psi)(z) = z + \sum_{n=2}^{\infty} \phi_n \psi_n z^n, \quad z \in U$$

For positive real parameters $p_1, A_1, \ldots, p_l, A_l$ and $q_1, B_1, \ldots, q_m, B_m$ $(l, m \in N = 1, 2, 3, \ldots)$ such that

$$1 + \sum_{n=1}^{m} B_m - \sum_{n=1}^{l} A_n \geq 0, \quad z \in U.$$  

The Wright generalized hypergeometric function (Wright, 1946)

$$\psi_m \left[ (p_1, A_1), \ldots, (p_l, A_l); (q_1, B_1), \ldots, (q_m, B_m); z \right] = \psi_m \left[ (p_n, A_n)_{1,l}, (q_n, B_n)_{1,m}; z \right]$$

is defined by:
\[
\Psi_m((p_n, A_n),_{n=1}^l (q_n, B_n),_{m=1}^l z) = \\
\sum_{n=0}^\infty \left\{ \prod_{i=0}^l (p_i + nA_i) \right\} \left\{ \prod_{j=0}^m (q_j + nB_j) \right\}^{-1} \frac{z^n}{n!}, \quad z \in U
\]

If \( A_t = 1(t = 1, 2, ..., l) \) and \( B_t = 1(t = 1, 2, ..., m) \) we have the relationship:

\[
\Psi_m((p_n, A_n),_{n=1}^l (q_n, B_n),_{m=1}^l z) = F_m(p_1, ..., p_l ; q_1, ..., q_m ; z)
\]

\( l \leq m + 1; l,m \in N \) is the general hypergeometric function (see for details (Wright, 1946)) where \( N \) denotes the set of all positive integers and \( (\lambda)_n \) is the Pochhammer symbol and

\[
\Omega = \prod_{l=0}^l (p_l) \prod_{m=0}^m (q_m) \]

By using the generalized hypergeometric function Dziok et al., (2003) introduced the linear operator. In 2004 Dziok et al. (2004) extended the linear operator by using Wright generalized hypergeometric function. First we define a function:

\[
\Phi_m((p_n, A_n),_{n=1}^l (q_n, B_n),_{m=1}^l z) := \Omega \Psi_m((p_n, A_n),_{n=1}^l (q_n, B_n),_{m=1}^l z)
\]

Let \( W[(p_n, A_n),_{n=1}^l (q_n, B_n),_{m=1}^l z] : A \to A \) be a linear operator defined by:

\[
W[(p_n, A_n),_{n=1}^l (q_n, B_n),_{m=1}^l z] f(z) = z \Omega \Phi_m((p_n, A_n),_{n=1}^l (q_n, B_n),_{m=1}^l z) \]

We observe that, for \( f(z) \) of the form (1.1), we have

\[
W[(p_n, A_n),_{n=1}^l (q_n, B_n),_{m=1}^l z] f(z) = z + \sum_{n=2}^\infty \sigma_n(p_1) z^n
\]

Where \( \sigma_n(p_1) \) is defined by

\[
\sigma_n(p_1) = \frac{\Omega \prod_{i=1}^l (p_i + A(n-1)) \cdot \Gamma(p_i + A(n-1))}{(n-1)! \prod_{j=1}^m (q_j + B(n-1)) \cdot \Gamma(q_j + B(n-1))}
\]

For convenience, we write:

\[
W[p_n, q_n] f(z) = W[p_1, A_1, ..., p_l, A_l ; q_1, B_1, ..., q_m, B_m] f(z)
\]

introduced by Dziok et al. (2004). In view of the relationship (3), the linear operator (6) includes the Dziok-Srivastava operator (Dziok et al., 2003), so that it includes (as its special cases) various other linear operators introduced and studied by Bernardi (1969), Carlson et al. (1984), Libera (1965), Livingston (1966), Rucheweyh (1975) and Srivastava et al. (1987).

Denoted by \( S(\alpha, \beta, \gamma, A, B) \), the subclass of \( A \) consisting of functions \( f(z) \) of the form (1) and satisfying the condition:

\[
\left| \frac{z W[p_1, q_1] f(z)}{W[p_1, q_1] f(z)} - 1 \right| < \beta, \quad z \in U
\]

Where \( W[p_1, q_1] f(z) \) is given by (8),

\[
0 \leq \alpha < 1, 0 < \beta \leq 1
\]

\[
\frac{B}{2(B-A)} < \gamma \leq \begin{cases} \frac{B}{2(B-A)\alpha} & , \alpha \neq 0 \\ 1 & , \alpha = 0 \end{cases}
\]

For fixed \(-1 \leq A \leq B \leq 1\) and \( 0 < B \leq 1 \). We also let

\[
TS^*(\alpha, \beta, \gamma, A, B) = S(\alpha, \beta, \gamma, A, B) \cap T
\]

Where

\[
T : \{ f \in A : f(z) = z - \sum_{n=2}^\infty \sigma_n(p_1) z^n, \quad z \in U \}
\]

A subclass of \( A \) introduced and studied by Silverman (1975).

By suitably specializing the values of \( A, B, l, m, p_1, ..., p_l, q_1, ..., q_m, A, B, \alpha, \beta \) and \( \gamma \), the class \( TS^*(\alpha, \beta, \gamma, A, B) \) leads to known subclasses studied in (Aghalary et al., 2002; Khairnar et al., 2008) and (Owa et al., 2002) and various new subclasses. In this paper we obtain sharp result for coefficient estimates, distortion theorem, radius of starlikeness and convexity and other related results.

For convenience we consider:
\[ \frac{B}{2(B - A)} < \gamma \leq \begin{cases} \frac{B}{2(B - A)\alpha} & \alpha \neq 0 \\ 1 & \alpha = 0 \end{cases} \]

For fixed \(-1 \leq A \leq B \leq 1\), \(0 \leq \alpha < 1, 0 < \beta \leq 1\) and \(0 < B \leq 1\), one or otherwise stated.

**CHARACTERIZATION PROPERTIES**

**Theorem 1**

Let the function \(f(z)\) be defined by (10) in the class \(TS^* (\alpha, \beta, \gamma, A, B)\) if and only if

\[ \sum_{n=2}^{\infty} 2\beta \gamma (B - A) (\alpha - n + (1 - B) (n - 1)) \sigma_n(p_1) a_n | \leq 2 \beta \gamma (1 - \alpha) (B - A) \]

Where \(\sigma_n(p_1)\) is given by (7).

**Proof**

Suppose,

\[ \sum_{n=2}^{\infty} 2\beta \gamma (B - A) (\alpha - n + (1 - B) (n - 1)) \sigma_n(p_1) a_n | \leq 2 \beta \gamma (1 - \alpha) (B - A) \]

We have

\[ |z(W[p, q], f(z)) - W[p, q], f(z)| - \beta 2\gamma (B - A) [z(W[p, q], f(z)) - W[p, q], f(z)] < 0 \]

With the provision

\[ |z| = r \leq 1; \text{ then the above condition bounded above by} \]

\[ \sum_{n=2}^{\infty} (n - 1) \sigma_n(p_1) a_n z^n | - \beta 2\gamma (B - A)(1 - \alpha) + \sum_{n=2}^{\infty} 2\gamma (B - A)(\alpha - n) + B(n - 1) \sigma_n(p_1) a_n z^n | < 0, \]

For \(|z| = r < 1\); then the above condition bounded above by

\[ \sum_{n=2}^{\infty} (n - 1) \sigma_n(p_1) a_n z^n | - 2 \beta \gamma (B - A)(\alpha - 1) - \beta \sum_{n=2}^{\infty} 2\gamma (B - A)(\alpha - n) + B(n - 1) \sigma_n(p_1) a_n z^n | < 0 \]

\[ \sum_{n=2}^{\infty} \frac{\sum_{n=2}^{\infty} 2\beta \gamma (B - A)(\alpha - n) + B(n - 1) \sigma_n(p_1) a_n z^n |}{z(W[p, q], f(z)) - W[p, q], f(z)} - \beta 2\gamma (B - A)(1 - \alpha) \]

Therefore \(TS^* (\alpha, \beta, \gamma, A, B)\). Conversely, Let

\[ \sum_{n=2}^{\infty} \frac{\sum_{n=2}^{\infty} 2\beta \gamma (B - A)(\alpha - n) + B(n - 1) \sigma_n(p_1) a_n z^n |}{z(W[p, q], f(z)) - W[p, q], f(z)} - \beta 2\gamma (B - A)(1 - \alpha) \leq 0 \]

As \(\text{Re}(z) \leq |z|\) for all \(z\), we have

\[ \text{Re} \left( \sum_{n=2}^{\infty} \frac{\sum_{n=2}^{\infty} 2\beta \gamma (B - A)(\alpha - n) + B(n - 1) \sigma_n(p_1) a_n z^n |}{z(W[p, q], f(z)) - W[p, q], f(z)} - \beta 2\gamma (B - A)(1 - \alpha) \right) < \beta \]

Choosing values of \(z\) on real axis such that \(z(W[p, q], f(z)) - W[p, q], f(z)\) is real and upon clearing the denominator through real values, and as \(z \to 1\) we obtain

\[ \sum_{n=2}^{\infty} 2\beta \gamma (B - A)(\alpha - n) + B(n - 1) \sigma_n(p_1) a_n | - 2 \beta \gamma (1 - \alpha) (B - A) \leq 0 \]

**Corollary 2**

Let the function \(f(z)\) defined by (1.10) be in the class \(TS^* (\alpha, \beta, \gamma, A, B)\). Then we have

\[ a_n \leq \frac{2 \beta \gamma (1 - \alpha) (B - A)}{2 \beta \gamma (B - A)(\alpha - n) + (1 - B) \beta (n - 1) \sigma_n(p_1)} \]

(12)

The equation (12) is attained for the function

\[ f(z) = -\frac{2 \beta \gamma (1 - \alpha) (B - A)}{2 \beta \gamma (B - A)(\alpha - n) + (1 - B) \beta (n - 1) \sigma_n(p_1)} z^n, \ (n \geq 2) \]

(13)

Where \(\sigma_n(p_1)\) is given by (7).

Let the functions \(f_j(z)(j = 1, 2)\) be defined by:
\[ f_j(z) = z - \sum_{n=1}^{\infty} a_{n,j} z^n \quad \text{for} \quad a_{n,j} \geq 0, \quad z \in U. \] (14)

**Theorem 2 (Closure theorem)**

Let the functions \( f_j(z) (j = 1, 2, \ldots, m) \) defined by (2.4) be in the classes \( TS^* (\alpha, \beta, \gamma, A, B) \) \((j = 1, 2, \ldots, m)\) respectively. Then the function \( h(z) \) defined by:

\[
h(z) = z - \frac{1}{m} \sum_{n=1}^{\infty} \left( \sum_{j=1}^{m} a_{n,j} \right) z^n
\]

Is in the class \( TS^* (\alpha, \beta, \gamma, A, B) \), where \( \alpha = \min \{\alpha_j\} \)

where \( 0 \leq \alpha_j \leq 1 \).

**Proof**

Since \( f_j \in TS^* (\alpha, \beta, \gamma, A, B) \), \((j = 1, 2, \ldots, m)\) by applying Theorem 1, to (4) we observe that

\[
\sum_{n=2}^{\infty} \frac{2\beta \gamma (B - A)(n - \alpha)}{2\beta \gamma (B - A)(n - \alpha) + (1 - B \beta)(n - 1)} \sigma_n(p_j) \frac{1}{m} \sum_{j=1}^{m} a_{n,j}
\]

\[
= \frac{1}{m} \sum_{j=1}^{m} \sum_{n=2}^{\infty} \frac{2\beta \gamma (B - A)(n - \alpha)}{2\beta \gamma (B - A)(n - \alpha) + (1 - B \beta)(n - 1)} \sigma_n(p_j) a_{n,j}
\]

\[
\leq \frac{1}{m} \sum_{j=1}^{m} 2\beta \gamma (1 - \alpha_j) (B - A) \leq 2\beta \gamma (1 - \alpha) (B - A)
\]

Which in view of Theorem 1, again implies that \( h \in TS^* (\alpha, \beta, \gamma, A, B) \) and so the proof is complete.

**Theorem 3 (Extreme points)**

Let

\[
f_i(z) = z \quad \text{and} \quad f_n(z) = z - \frac{2\beta \gamma (1 - \alpha)(B - A)}{2\beta \gamma (B - A)(n - \alpha) + (1 - B \beta)(n - 1)} \sigma_n(p_i) z^n, \quad (n \geq 2)
\] (15)

where \( \sigma_n(p_i) \) is given by (7). Then \( f(z) \) is in the class \( TS^* (\alpha, \beta, \gamma, A, B) \) if and only if it can be expressed in the form:

\[
f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z)
\] (16)

Where \( \mu_n \geq 0 \) \((n \geq 1)\) and \( \sum_{n=1}^{\infty} \mu_n = 1 \).

**Proof**

Suppose that

\[
f(z) = \mu_1 f_1(z) + \sum_{n=2}^{\infty} \mu_n f_n(z)
\]

\[
= \mu_1 z + \sum_{n=2}^{\infty} \mu_n \left[ z - \frac{2\beta \gamma (1 - \alpha)(B - A)}{2\beta \gamma (B - A)(n - \alpha) + (1 - B \beta)(n - 1)} \sigma_n(p_i) z^n \right]
\]

\[
= \mu_1 z + \sum_{n=2}^{\infty} \mu_n z - \sum_{n=2}^{\infty} \mu_n \left[ \frac{2\beta \gamma (1 - \alpha)(B - A)}{2\beta \gamma (B - A)(n - \alpha) + (1 - B \beta)(n - 1)} \sigma_n(p_i) \right] z^n
\]

\[
= \mu_1 z + \sum_{n=2}^{\infty} \mu_n z - \left[ \frac{2\beta \gamma (1 - \alpha)(B - A)}{2\beta \gamma (B - A)(n - \alpha) + (1 - B \beta)(n - 1)} \sigma_n(p_i) \right] \mu_n z^n.
\]

Then it follows that

\[
\sum_{n=2}^{\infty} \mu_n z = \sum_{n=2}^{\infty} \mu_n z - \left[ \frac{2\beta \gamma (1 - \alpha)(B - A)}{2\beta \gamma (B - A)(n - \alpha) + (1 - B \beta)(n - 1)} \sigma_n(p_i) \right] \mu_n z^n.
\]

By Theorem 1, \( f \in TS^* (\alpha, \beta, \gamma, A, B) \).

Conversely, suppose that \( f \in TS^* (\alpha, \beta, \gamma, A, B) \). Then

\[
a_n \leq \frac{2\beta \gamma (1 - \alpha)(B - A)}{2\beta \gamma (B - A)(n - \alpha) + (1 - B \beta)(n - 1)} \sigma_n(p_i) \quad (n \geq 2).
\]

We set

\[
\mu_n = \frac{2\beta \gamma (B - A)(n - \alpha) + (1 - B \beta)(n - 1)}{2\beta \gamma (B - A)(n - \alpha) + (1 - B \beta)(n - 1)} \sigma_n(p_i) a_n \quad (n \geq 2)
\]

and \( \mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n \). Then using (1.10) we obtain

\[
f(z) = z - \sum_{n=2}^{\infty} \mu_n z^n
\]

\[
= z - \sum_{n=2}^{\infty} \mu_n \left[ z - \frac{2\beta \gamma (1 - \alpha)(B - A)}{2\beta \gamma (B - A)(n - \alpha) + (1 - B \beta)(n - 1)} \sigma_n(p_i) z^n \right]
\]

\[
= z - \sum_{n=2}^{\infty} \mu_n \left[ z - f_n(z) \right]
\]
\[ z = \sum_{n=2}^{\infty} \frac{\mu_n z}{n} + \sum_{n=2}^{\infty} \frac{\mu_n f_n(z)}{n} \]
\[ = \mu_1 f_1(z) + \sum_{n=2}^{\infty} \mu_n f_n(z) \]
\[ = \sum_{n=1}^{\infty} \mu_n f_n(z). \]

This completes the proof of Theorem 3.

**DISTORTION BOUNDS**

**Theorem 4**

Let the function \( f(z) \) defined by (1.10) belong to \( TS^*(\alpha, \beta, \gamma, A, B) \). Then

\[ |f(z)| \geq |z| \left\{ 1 - \frac{2\beta(1-\alpha)(B-A)}{1 + 2\beta(1-\alpha)(B-A)} \right\} \]

and

\[ |f(z)| \leq |z| \left\{ 1 + \frac{2\beta(1-\alpha)(B-A)}{1 + 2\beta(1-\alpha)(B-A)} \right\}, \]

where \( \sigma_2(p_1) \) is given by (7).

**Proof**

In the view of (11) and the fact that \( \sigma_n(p_1) \) is non-decreasing for \( n \geq 2 \), we have;

\[ [2\beta(1-\alpha)(B-A) + (1-B\beta)\sigma_2(p_1)] \sum_{n=2}^{\infty} a_n \leq [2\beta(1-\alpha)(B-A) + (1-B\beta)(n-1)\sigma_2(p_1)] \sum_{n=2}^{\infty} a_n \]

which is equivalent to,

\[ \sum_{n=2}^{\infty} a_n \leq \frac{2\beta(1-\alpha)(B-A)}{1 + 2\beta(1-\alpha)(B-A)} \]

Using (10) and (19), we obtain

\[ |f(z)| \geq |z| \left\{ 1 - \frac{2\beta(1-\alpha)(B-A)}{1 + 2\beta(1-\alpha)(B-A)} \right\} \]

and

\[ |f(z)| \leq |z| \left\{ 1 + \frac{2\beta(1-\alpha)(B-A)}{1 + 2\beta(1-\alpha)(B-A)} \right\}. \]

Hence the proof is complete.

**RADIUS OF STARLIKENESS AND CONVEXITY**

Next we obtain the radii of close-to-convexity, starlikeness and convexity for the class \( TS^*(\alpha, \beta, \gamma, A, B) \).

**Theorem 5**

Let the function \( f(z) \) defined by (10) belong to the class \( TS^*(\alpha, \beta, \gamma, A, B) \).

Then \( f(z) \) is close-to-convex of order \( \eta \) \( (0 \leq \eta < 1) \) in the disc \( |z| < r_1 \), where

\[ r_1 := \inf_{n=2} \left\{ \frac{(1-\eta)[2\beta(1-\alpha)(n+\alpha) + (1-B\beta)(n-1)]\sigma_n(p_1)}{2n\beta(1-\alpha)(B-A)} \right\}^{1/1}. \]

(20)

Where \( \sigma_n(p_1) \) is given by (1.7). The result is sharp, with external function \( f(z) \) given by (15).

**Proof**

Given \( f \in T \), and \( f \) is close-to-convex of order \( \eta \), we have

\[ |f'(z)| - 1 < 1 - \eta. \] (21)

For the left hand side of (21) we have

\[ |f'(z)| - 1 < \sum_{n=2}^{\infty} a_n |z|^{n-1}. \]

The last expression is less than \( 1 - \sigma \) if

\[ \sum_{n=2}^{\infty} a_n |z|^{n-1} < 1. \]

Using the fact, that \( f \in TS^*(\alpha, \beta, \gamma, A, B) \) if and only if

\[ \sum_{n=2}^{\infty} \frac{2\beta(1-\alpha)(n+\alpha) + (1-B\beta)(n-1)}{2\beta(1-\alpha)(B-A)} \sigma_n(p_1) a_n < 1. \]
We can say (21) is true if
\[
\frac{n}{1-\eta} |z|^{n-1} < \frac{2B\gamma(B-A)(n-\alpha) + (1-B\beta)(n-1)\sigma_{n}(p_1)}{2B\gamma(1-\alpha)(B-A)}.
\]
or, equivalently,
\[
|z|^{n-1} < \left(\frac{1-\eta}{n-\eta}\right)\frac{2B\gamma(B-A)(n-\alpha) + (1-B\beta)(n-1)\sigma_{n}(p_1)}{2B\gamma(1-\alpha)(B-A)}.
\]
where \(\sigma_{n}(p_1)\) is given by (7). This completes the proof.

**Theorem 6**

Let \(f \in TS^{*}(\alpha, \beta, \gamma, A, B)\). Then \(f(z)\) is starlike of order \(\eta(0 \leq \eta < 1)\) in the disc \(|z| < r_2\), that is,
\[
\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \eta, \quad (1 < r_2; 0 \leq \eta < 1),
\]
where
\[
r_2 = \inf_{n \geq 2} \left[\left(\frac{1-\eta}{n-\eta}\right)\frac{2B\gamma(B-A)(n-\alpha) + (1-B\beta)(n-1)\sigma_{n}(p_1)}{2B\gamma(1-\alpha)(B-A)}\right]^{1/(n-1)}.
\]
(22)

\(f(z)\) is convex of order \(\eta(0 \leq \eta < 1)\) in the disc \(|z| < r_3\), that is, \(\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \eta, \quad (1 < r_3; 0 \leq \eta < 1)\), where;
\[
r_3 = \inf_{n \geq 2} \left[\left(\frac{1-\eta}{n(n-\eta)}\right)\frac{2B\gamma(B-A)(n-\alpha) + (1-B\beta)(n-1)\sigma_{n}(p_1)}{2B\gamma(1-\alpha)(B-A)}\right]^{1/(n-1)}.
\]
(23)

Where \(\sigma_{n}(p_1)\) is given by (7). Each of these results are sharp for the external function \(f(z)\) given by (15).

**Proof**

Given \(f \in T\), and \(f\) is starlike of order \(\eta\), we have
\[
\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \eta.
\]
(24)

For the left hand side of (24) we have
\[
\left|\frac{zf'(z)}{f(z)} - 1\right| < \sum_{n=2}^{\infty} (n-1)a_n |z|^{n-1}.
\]

The last expression is less than \(1 - \eta\) if
\[
\sum_{n=2}^{\infty} a_n |z|^{n-1} < 1.
\]

Using the fact, that \(f \in TS^{*}(\alpha, \beta, \gamma, A, B)\) if and only if
\[
\sum_{n=2}^{\infty} \left[\frac{2B\gamma(B-A)(n-\alpha) + (1-B\beta)(n-1)\sigma_{n}(p_1)}{2B\gamma(1-\alpha)(B-A)}\right] a_n \leq 1.
\]
We can say (24) is true if
\[
\frac{n}{n-\eta} |z|^{n-1} < \frac{2B\gamma(B-A)(n-\alpha) + (1-B\beta)(n-1)\sigma_{n}(p_1)}{2B\gamma(1-\alpha)(B-A)}.
\]

or, equivalently,
\[
|z|^{n-1} < \left(\frac{1-\eta}{n-\eta}\right)\frac{2B\gamma(B-A)(n-\alpha) + (1-B\beta)(n-1)\sigma_{n}(p_1)}{2B\gamma(1-\alpha)(B-A)}.
\]

This yields the starlikeness of the family which completes the proof.

(ii) Using the fact that \(f(z)\) is convex if and only if \(zf''(z)\) is starlike, we can prove (ii), on lines similar to the proof of (i).

**MODIFIED HADAMARD PRODUCTS**

Let the functions \(f_1(z)(j=1,2)\) be defined by (14). The modified Hadamard product of \(f_1(z)\) and \(f_2(z)\) is defined by:
\[
(f_1 \ast f_2)(z) = z - \sum_{n=2}^{\infty} a_{n,j}a_{n,2} z^{n}.
\]
Using the techniques of Schild et al. (1975), we prove the following results.

**Theorem 7**

For functions \(f_j(z)(j=1,2)\) defined by (14), let \(f_1 \in TS^{*}(\alpha, \beta, \gamma, A, B)\) and \(f_2 \in TS^{*}(\mu, \beta, \gamma, A, B)\). Then \((f_1 \ast f_2) \in TS^{*}(\xi, \beta, \gamma, A, B)\), where
\[
\xi = 1 - \frac{2B\gamma(B-A)(\xi-\alpha)(\xi-\mu)(1+2B\gamma(B-A)-B\beta)}{\Lambda_1(\alpha, \beta, \gamma, A, B, 2)\Lambda_2(\mu, \beta, \gamma, A, B, 2)\sigma_{n}(p_1)-4B\gamma^2(1-\alpha)(B-A)^2(1-\mu)}.
\]
(25)
and

\[\Lambda_1(\alpha, \beta, \gamma, A, B, 2) = 2\beta\gamma(B - A)(2 - \alpha) + (1 - B\beta)\]

\[\Lambda_2(\mu, \beta, \gamma, A, B, 2) = 2\beta\gamma(B - A)(2 - \mu) + (1 - B\beta)\]

where \(\sigma_z(p_1)\) is given by (7).

**Proof**

In view of Theorem 1, it suffice to prove that

\[\sum_{n=2}^{\infty} \frac{[2\beta\gamma(B - A)(n - \xi) + (1 - B\beta)(n - 1)]\sigma_z(p_1)}{2\beta\gamma(1 - \xi)(B - A)} a_n a_{n+2} \leq 1, \quad (0 \leq \xi < 1)\]

Where \(\xi\) is defined by (25). On the other hand, under the hypothesis, it follows from (11) and the Cauchy-Schwarz inequality that

\[\sum_{n=2}^{\infty} \frac{[\Lambda_1(\alpha, \beta, \gamma, A, B, n)]^{1/2} [\Lambda_2(\mu, \beta, \gamma, A, B, n)]^{1/2} \sigma_z(p_1)}{\sqrt{(1 - \alpha)(1 - \mu)}} \leq 1,\]

(26)

where

\[\Lambda_1(\alpha, \beta, \gamma, A, B, n) = 2\beta\gamma(B - A)(n - \alpha) + (1 - B\beta)(n - 1)\]

\[\Lambda_2(\mu, \beta, \gamma, A, B, n) = 2\beta\gamma(B - A)(n - \mu) + (1 - B\beta)(n - 1)\]

(27)

Thus we need to find the largest \(\xi\) such that

\[\sum_{n=2}^{\infty} \frac{2\beta\gamma(B - A)(n - \xi) + (1 - B\beta)(n - 1)]\sigma_z(p_1)}{2\beta\gamma(1 - \xi)(B - A)} a_n a_{n+2} \leq 1,\]

or, equivalently that:

\[\sqrt{a_n a_{n+2}} \leq \frac{1 - \xi}{\sqrt{(1 - \alpha)(1 - \mu)}} \frac{\sqrt{\Lambda_1(\alpha, \beta, \gamma, A, B, n)}}{2\beta\gamma(B - A)(n - \xi) + (1 - B\beta)(n - 1)},\]

(n \geq 2).

In view of (26) it is sufficient to find largest \(\xi\) such that

\[\frac{2\beta\gamma(B - A)\sqrt{(1 - \alpha)(1 - \mu)}(\sigma_z(p_1))^{1/2}}{[\Lambda_1(\alpha, \beta, \gamma, A, B, n)]^{1/2} [\Lambda_2(\mu, \beta, \gamma, A, B, n)]^{1/2}} \leq \frac{1 - \xi}{\sqrt{(1 - \alpha)(1 - \mu)}} \frac{[\Lambda_1(\alpha, \beta, \gamma, A, B, n)]^{1/2} [\Lambda_2(\mu, \beta, \gamma, A, B, n)]^{1/2}}{2\beta\gamma(B - A)(n - \xi) + (1 - B\beta)(n - 1)},\]

which yields

\[\xi = \Psi(n) = 1 - \varepsilon \]

\[\frac{2\beta\gamma(B - A)(1 - \alpha)[1 - B\beta](n - 1) + 2\beta\gamma(B - A) - 2\beta\gamma(B - A) - 2\beta\gamma(B - A)}{[\Lambda_1(\alpha, \beta, \gamma, A, B, n)] [\Lambda_2(\mu, \beta, \gamma, A, B, n)] \sigma_z(p_1) - 4\beta\gamma(B - A)^2 (1 - \alpha)}\]

(28)

for \(n \geq 2\) is an increasing function of \(n (n \geq 2)\) and letting \(n = 2\) in (29), we have

\[\xi = \Psi(2) = 1 - \varepsilon \]

\[\frac{2\beta\gamma(B - A)(1 - \alpha)[1 - B\beta]2 + 2\beta\gamma(B - A) - 2\beta\gamma(B - A) - 2\beta\gamma(B - A)}{[\Lambda_1(\alpha, \beta, \gamma, A, B, 2)] [\Lambda_2(\mu, \beta, \gamma, A, B, 2)] \sigma_z(p_1) - 4\beta\gamma(B - A)^2 (1 - \alpha)}\]

Where \([\Lambda_1(\alpha, \beta, \gamma, A, B, 2)]\) and \([\Lambda_2(\mu, \beta, \gamma, A, B, 2)]\) as defined in (27), where \(\sigma_2(p_1)\) is given by (7).

**Theorem 8**

Let the functions \(f_j(z) (j = 1, 2)\) be in the class \(TS^*(\alpha, \beta, \gamma, A, B)\). Then \((f_1 \ast f_2) \in TS^*(\rho, \beta, \gamma, A, B)\), where \(\sigma_2(p_1)\) is given by (7).

**Proof**

By taking \(\mu = \alpha\), in the above theorem, the result follows.

**Theorem 9**

Let the function \(f(z)\) be in the class \(TS^*(\alpha, \beta, \gamma, A, B)\). Also let \(g(z) = z - \sum_{n=2}^{\infty} b_n z^n\) for \(|b_n| \leq 1\).

Then \((f \ast g) \in TS^*(\alpha, \beta, \gamma, A, B)\).

**Proof**

Since

\[\sum_{n=2}^{\infty} [2\beta\gamma(B - A)(n - \alpha) + (1 - B\beta)(n - 1)]\sigma_z(p_1) a_n b_n \leq \sum_{n=2}^{\infty} [2\beta\gamma(B - A)(n - \alpha) + (1 - B\beta)(n - 1)] \sigma_z(p_1) a_n b_n \leq 2\beta\gamma(1 - \alpha)(B - A)\]
it follows that \((f \ast g) \in TS^*(\alpha, \beta, \gamma, A, B)\), by the view of Theorem 1.

**Theorem 10**

Let the functions \(f_j(z) (j = 1, 2)\) defined by (14), be in the class \(TS^*(\alpha, \beta, \gamma, A, B)\). Then the function \(h(z)\) defined by \(h(z) = z - \sum_{n=2}^\infty (a^2_{n,1} + a^2_{n,2})z^n\) is in the class \(TS^*(\xi, \beta, \gamma, A, B)\) where

\[
\xi = 1 - \frac{4\beta\gamma(B - A)(1 - \alpha)^2}{\sigma_z(p_i)[1 + 2\beta\gamma(B - A)(2 - \alpha) - B\beta]^2 - 8\beta^2\gamma^2(1 - \alpha)^2(B - A)^2},
\]

where \(\sigma_z(p_i)\) is given by (7).

**Proof**

In view of Theorem 1, it suffice to prove that

\[
\sum_{n=2}^\infty \left[\frac{2\beta\gamma(B - A)(n - \xi) + (1 - B\beta)(n - 1)}{2\beta\gamma(1 - \xi)(B - A)}\right] \left(\sigma_z(p_i)\right) (a^2_{n,1} + a^2_{n,2}) \leq 1,
\]

where \(f_j \in TS^*(\alpha, \beta, \gamma, A, B)\) we find from (2.4) and Theorem 1, that:

\[
\sum_{n=2}^\infty \left[\frac{2\beta\gamma(B - A)(n - \xi) + (1 - B\beta)(n - 1)}{2\beta\gamma(1 - \xi)(B - A)}\right] a^2_{n,j} \leq 1
\]

(29)

this yields

\[
\sum_{n=2}^\infty \left[\frac{2\beta\gamma(B - A)(n - \alpha) + (1 - B\beta)(n - 1)}{2\beta\gamma(1 - \alpha)(B - A)}\right] \left(\sigma_z(p_i)\right) a_{n,j} \leq 1,
\]

(30)

On comparing (30) and (31), it is easily seen that the inequality (29) will be satisfied if

\[
\frac{2\beta\gamma(B - A)(n - \xi) + (1 - B\beta)(n - 1)}{2\beta\gamma(1 - \xi)(B - A)} \leq \frac{1}{2} \left(\frac{2\beta\gamma(B - A)(n - \alpha) + (1 - B\beta)(n - 1)}{2\beta\gamma(1 - \alpha)(B - A)}\right)^2, \quad n \geq 2.
\]

That is

\[
\xi = 1 - \frac{4\beta\gamma(B - A)(1 - \alpha)^2(n - 1)(1 + 2\beta\gamma(B - A) - B\beta)}{\sigma_z(p_i)[2\beta\gamma(B - A)(n - \alpha) + (1 - B\beta)(n - 1)] - 8\beta^2\gamma^2(1 - \alpha)^2(B - A)^2},
\]

(31)

since

\[
\Psi(n) = 1 - \frac{4\beta\gamma(B - A)(1 - \alpha)^2(n - 1)(1 + 2\beta\gamma(B - A) - B\beta)}{\sigma_z(p_i)[1 + 2\beta\gamma(B - A)(2 - \alpha) - B\beta]^2 - 8\beta^2\gamma^2(1 - \alpha)^2(B - A)^2}
\]

is an increasing function of \(n\) \((n \geq 2)\). Taking \(n = 2\) in (32), we have:

\[
\xi = \Psi(2) = 1 - \frac{4\beta\gamma(B - A)(1 - \alpha)^2}{\sigma_z(p_i)[1 + 2\beta\gamma(B - A)(2 - \alpha) - B\beta]^2 - 8\beta^2\gamma^2(1 - \alpha)^2(B - A)^2},
\]

this completes the proof.

**INCLUSION RELATIONS INVOLVING N \(\delta (E)\)**

Following (Goodman, 1957; Rucheweyh, 1981), we define the \(\delta\) - neighbourhood of function \(f \in T\), by

\[
N\_\delta(f) = \left\{ h \in T : h(z) = z - \sum_{n=2}^\infty b_n z^n \quad \text{and} \quad \sum_{n=2}^\infty |b_n| \leq \delta \right\}.
\]

(32)

Particularly for the identity function \(e(z) = z\), we have

\[
N\_\delta(e) = \left\{ h \in T : h(z) = z - \sum_{n=2}^\infty b_n z^n \quad \text{and} \quad \sum_{n=2}^\infty |b_n| \leq \delta \right\}.
\]

(33)

Now we obtain inclusion relations of the class \(TS^*(\alpha, \beta, \gamma, A, B)\).

**Theorem 11**

If

\[
\delta := \frac{4\beta\gamma(B - A)(1 - \alpha)}{\sigma_z(p_i)[1 + 2\beta\gamma(B - A)(2 - \alpha) - B\beta]},
\]

(34)

where \(\sigma_z(\alpha_1)\) is given by (7). Then \(TS^*(\alpha, \beta, \gamma, A, B) \subset N\_\delta(e)\).

**Proof**

For \(f \in TS^*(\alpha, \beta, \gamma, A, B)\), Theorem 2.1 immediately yields
Then, from (11) and (36) that

\[ \sum_{n=2}^{\infty} a_n \leq \frac{2\beta\gamma(1-\alpha)(B-A)}{2\beta\gamma(B-A)(2-\alpha) + (1-B\beta)} \sigma_2(p_1). \]  

(35)

On the other hand, from (11) and (36) that

\[ \sum_{n=2}^{\infty} a_n \leq \frac{2\beta\gamma(1-\alpha)(B-A)}{2\beta\gamma(B-A)(2-\alpha) + (1-B\beta)} \sigma_2(p_1). \]

so that

\[ \sum_{n=2}^{\infty} a_n \leq \frac{2\beta\gamma(1-\alpha)(B-A)}{2\beta\gamma(B-A)(2-\alpha) + (1-B\beta)} \sigma_2(p_1). \]

(36)

which, in view of (33) completes the proof of Theorem 11.

Next we determine the neighborhood for the class

\[ TS^*(\rho)(\alpha, \beta, \gamma, A, B) \]

which, in view of (33) completes the proof of Theorem 11.

Next, since \( h \in TS^*(\rho)(\alpha, \beta, \gamma, A, B) \), we have

\[ \sum_{n=2}^{\infty} b_n \leq \frac{2\beta\gamma(1-\alpha)(B-A)}{1 + 2\beta\gamma(B-A)(2-\alpha) - B\beta} \sigma_2(p_1). \]

so that

\[ f(z) - 1 < \frac{\sum_{n=2}^{\infty} a_n - b_n}{1 - \sum_{n=2}^{\infty} b_n} \leq \frac{\delta}{2}. \]

provided that \( \rho \) is given precisely by (40). Thus by definition, \( f \in TS^{*(\rho)}(\alpha, \beta, \gamma, A, B) \) for \( \rho \) given by (40), which completes the proof.

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