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Full Length Research Paper

Interval cut-set of interval-valued intuitionistic fuzzy sets

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In this paper, different types of interval cut-set of interval-valued intuitionistic fuzzy sets (IVIFSs), complement of these cut-sets are defined. Some properties of those cut-set of IVIFSs are investigated. Also three decomposition theorems of IVIFSs are defined. These works can also be used in setting up the basic theory of IVIFSs.

Key words: Interval-valued intuitionistic fuzzy sets, interval cut-set on interval-valued intuitionistic fuzzy sets, decomposition theorem.

INTRODUCTION

In 1965, Zadeh introduced the concept of fuzzy subsets. Latter many authors defined different directions of fuzzy subsets. Turksen (1986) generalized the concept of fuzzy set in terms of interval-valued fuzzy set (IVFS). Several researchers present a number of results using IVFSs. Using these concept of (IVFS) Pal and Shyamal (2006) introduced interval-valued fuzzy matrices and shown several properties of them. Atanassov (1986, 1989, 1994) introduced the concept of intuitionistic fuzzy sets (IFSs), which is more generalization of fuzzy subsets and as well as IVFSs. Several authors present a number of results using IFSs. By the concept of IFSs, first time Pal (2001) introduced intuitionistic fuzzy determinant. Latter on Pal, Khan and Shyamal (2002), introduced intuitionistic fuzzy matrices and distance between intuitionistic fuzzy matrices. Recently, Bhowmik and Pal (2008, 2009) introduced some results on intuitionistic fuzzy matrices, intuitionistic circulant fuzzy matrices and generalized intuitionistic fuzzy matrices. After the work of Atanassov (1986), again Gargo and Atanassov (1989) introduced the interval-valued intuitionistic fuzzy sets (IVIFSs). They have shown several properties on IVIFSs and shown some applications of IVIFSs. Mondal and Samanta (2002) introduced an another concept of IFSs called generalized IFSs. Jana and Pal (2006) studied some operators defined over IVIFS. Bhowmik and Pal (2010, 2009) defined generalized interval-valued intuitionistic fuzzy set (GIVIFS) and presented various properties of it.

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Preliminaries

The concept of interval arithmetics are recalled. Let $[I]$ be the set of all closed subintervals of the interval $[0,1]$. An interval on $[I]$, say $\overline{a}$, is a closed subinterval of $[I]$ i.e., $\overline{a}=[a^-,a^+]$ where a^- and a^+ are lower and upper limits of $\overline{a}$ respectively and satisfy the condition $0 \le a^- \le a^+ \le 1$. For any two interval $\overline{a}$ and $\overline{b}$ where $\overline{a}=[a^-,a^+]$ and $\overline{b}=[b^-,b^+]$ then (i) $\overline{a}=\overline{b} \Leftrightarrow a^-=b^-$, $a^+=b^+$, (ii) $\overline{a}\le\overline{b} \Leftrightarrow a^- \le b^-$, $a^+ \le b^+$ and (iii) $\overline{a}<\overline{b} \Leftrightarrow a^- < b^-$, $a^+ < b^+$ and $\overline{a}\ne\overline{b}$.

Definition 1: An IVIFS A over X (universe of discourse) is an object having the form $A=\{\langle x, M_A(x), N_A(x)\rangle \mid x \in X\}$, where $M_A(x): X \to [I]$ and $N_A(x): X \to [I]$. The intervals $M_A(x)$ and $N_A(x)$ denote the intervals of the degree of membership and degree of non-membership of the element x to the set A, where

$$M_A(x) = [M_{AL}(x), M_{AU}(x)]$$

and $N_A(x) = [N_{AL}(x), N_{AU}(x)]$,

for all $x \in X$, with the condition $0 \le M_{AU}(x) + N_{AU}(x) \le 1$. For simplicity, we denot $A = \{\langle x, [A^-(x), A^+(x)], [B^-(x), B^+(x)]\rangle \mid x \in X\}$. Let $\Phi(X)$ be the set of all IVIFSs defined on X.

Some operations on IVIFSs

In 2010, Bhowmik and Pal defined some relational operations on IVIFSs. Let A and B be two IVIFSs on X, where

$$A = \{\langle [M_{AL}(x), M_{AU}], [N_{AL}(x), N_{AU}(x)] : x \in X\rangle\} and$$

$$B = \{\langle [M_{BL}(x), M_{BU}], [N_{BL}(x), N_{BU}(x)] : x \in X\rangle\}.$$

$$Then (1) A = B \Leftrightarrow M_A(x) = M_B(x) \; and \; N_A(x) = N_B(x), for all \; x \in X.$$

$$(2) A \subseteq B \; iff \; \{(M_{AU}(x) \le M_{BU}(x) \; and M_{AL}(x) \le M_{BL}(x))\} and \; \{(N_{AU}(x) \ge N_{BU}(x) \; and \; N_{AL}(x) \ge N_{BL}(x))\}, for \; all: x \in X.$$

$$(3) \; \overline{A} = \{\langle x, N_A(x), M_A(x)\rangle \mid x \in X\}, for \; all: x \in X.$$

$$(4) A \cap B = \{\langle [min\{M_{AL}(x), M_{BL}(x)\}, min\{M_{AU}(x), M_{BU}(x)\}], [max\{N_{AL}(x), N_{BL}(x)\}, max\{N_{AU}(x), N_{BU}(x)\}]\rangle : x \in X\}.$$

$$(5) A \cup B = \{\langle [max\{M_{AL}(x), M_{BL}(x)\}, max\{M_{AU}(x), M_{BU}(x)\}], [min\{N_{AL}(x), N_{BL}(x)\}, min\{N_{AU}(x), N_{BU}(x)\}]\rangle : x \in X\}.$$

Interval cut-sets on IVIFS and some results

In (2005), Wang and Jin has introduced some kinds of cut-sets for interval-valued fuzzy sets based on fuzzy interval and interval-valued fuzzy sets. Here we define some types of interval cut-sets for IVIFS.

Definition 2: *Let A be an IVIFS and $\alpha = [\alpha_1, \alpha_2]$, $\beta = [\beta_1, \beta_2] \in [I]$.*

Then different types of interval cut-sets on IVIFS A are defined as follows:

$$A^{(1,1)}_{\alpha\beta} = A^{(1,1)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]} = \{\langle x, [A^-(x) \ge \alpha_1, A^+(x) \ge \alpha_2], [B^-(x) \ge \beta_1, B^+(x) \ge \beta_2]\rangle \mid x \in X\}$$

$$A^{(1,2)}_{\alpha\beta} = A^{(1,2)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]} = \{\langle x, [A^-(x) \ge \alpha_1, A^+(x) > \alpha_2], [B^-(x) \ge \beta_1, B^+(x) > \beta_2]\rangle \mid x \in X\},$$

$$A^{(2,1)}_{\alpha\beta} = A^{(2,1)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]} = \{\langle x, [A^-(x) > \alpha_1, A^+(x) \ge \alpha_2], [B^-(x) > \beta_1, B^+(x) \ge \beta_2]\rangle \mid x \in X\},$$

$$A^{(2,2)}_{\alpha\beta} = A^{(2,2)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]} = \{\langle x, [A^-(x) > \alpha_1, A^+(x) > \alpha_2], [B^-(x) > \beta_1, B^+(x) > \beta_2]\rangle \mid x \in X\},$$

$$A^{(3,3)}_{\alpha\beta} = A^{(3,3)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]} = \{\langle x, [A^-(x) \le \alpha_1, A^+(x) \le \alpha_2], [B^-(x) \le \beta_1, B^+(x) \le \beta_2]\rangle \mid x \in X\},$$

$$A^{(3,4)}_{\alpha\beta} = A^{(3,4)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]} = \{\langle x, [A^-(x) \le \alpha_1, A^+(x) < \alpha_2], [B^-(x) \le \beta_1, B^+(x) < \beta_2]\rangle \mid x \in X\},$$

$$A^{(4,3)}_{\alpha\beta} = A^{(4,3)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]} = \{\langle x, [A^-(x) < \alpha_1, A^+(x) \le \alpha_2], [B^-(x) < \beta_1, B^+(x) \le \beta_2]\rangle \mid x \in X\},$$

$$A^{(4,4)}_{\alpha\beta} = A^{(4,4)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]} = \{\langle x, [A^-(x) < \alpha_1, A^+(x) < \alpha_2], [B^-(x) < \beta_1, B^+(x) < \beta_2]\rangle \mid x \in X\}.$$

where $A^{(i,j)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}$ is the (i,j) th $([\alpha_1,\alpha_2],[\beta_1,\beta_2])$ interval cut-set of IVIFS A.

In the following we discussed some propositions for IVIFS and interval cut-sets of IVIFSs.

Definition 3: *Let X be a nonempty set, A be an IVIFS on $\Phi(X)$ and $\alpha, \beta \in [I]$, where $\alpha = [\alpha_1, \alpha_2]$, $\beta = [\beta_1, \beta_2]$. We define*

$$(A^-)^1_{\alpha_1,\beta_1} = \{\langle x, [A^-(x) \ge \alpha_1, A^+(x)], [B^-(x) \ge \beta_1, B^+(x)]\rangle \mid x \in X\},$$

$$(A^-)^2_{\alpha_1,\beta_1} = \{\langle x, [A^-(x) > \alpha_1, A^+(x)], [B^-(x) > \beta_1, B^+(x)]\rangle \mid x \in X\},$$

$$(A^+)^1_{\alpha_2,\beta_2} = \{\langle x, [A^-(x), A^+(x) \ge \alpha_2], [B^-(x), B^+(x) \ge \beta_2]\rangle \mid x \in X\},$$

$$(A^+)^2_{\alpha_2,\beta_2} = \{\langle x, [A^-(x), A^+(x) > \alpha_2], [B^-(x), B^+(x) > \beta_2]\rangle \mid x \in X\},$$

$$(A^-)^3_{\alpha_1,\beta_1} = \{\langle x, [A^-(x) \le \alpha_1, A^+(x)], [B^-(x) \le \beta_1, B^+(x)]\rangle \mid x \in X\},$$

$$(A^-)^4_{\alpha_1,\beta_1} = \{\langle x, [A^-(x) < \alpha_1, A^+(x)], [B^-(x) < \beta_1, B^+(x)]\rangle \mid x \in X\},$$

$$(A^+)^3_{\alpha_2,\beta_2} = \{\langle x, [A^-(x), A^+(x) \le \alpha_2], [B^-(x), B^+(x) \le \beta_2]\rangle \mid x \in X\},$$

$$(A^+)^4_{\alpha_2,\beta_2} = \{\langle x, [A^-(x), A^+(x) < \alpha_2], [B^-(x), B^+(x) < \beta_2]\rangle \mid x \in X\}.$$

Proposition 1: Let X be a nonempty set, A is an IVIFS and $\alpha=[\alpha_1,\alpha_2]$, $\beta=[\beta_1,\beta_2]\in[I]$, then

$$A^{(i,j)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}=(A^-)^i_{\alpha_1,\beta_1}\cap(A^+)^j_{\alpha_2,\beta_2}\; i,j=1,2.$$

$$\text{and}\; A^{(i,j)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}=(A^-)^i_{\alpha_1,\beta_1}\cup(A^+)^j_{\alpha_2,\beta_2}\; i,j=3,4.$$

Proof : Here we prove only for $i=1, j=1$ and $\alpha=[\alpha_1,\alpha_2]$, $\beta=[\beta_1,\beta_2]\in[I]$ and other proves are similar.

$$A^{(1,1)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}=(A^-)^1_{\alpha_1,\beta_1}\cap(A^+)^1_{\alpha_2,\beta_2}$$

Let $x\in A^{(1,1)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}$

$$\Leftrightarrow\{x\in X:\{\langle x,[A^-(x)\geq\alpha_1,A^+(x)\geq\alpha_2],[B^-(x)\geq\beta_1,B^+(x)\geq\beta_2]\rangle\}$$

$$\Leftrightarrow\{\langle x,[A^-(x)\geq\alpha_1,A^+(x)],[B^-(x)\geq\beta_1,B^+(x)]\rangle\mid x\in X\}$$
and
$$\{\langle x,[A^-(x),A^+(x)\geq\alpha_2],[B^-(x),B^+(x)\geq\beta_2]\rangle\mid x\in X\}$$

$$\Leftrightarrow x\in(A^-)^i_{\alpha_1,\beta 1}\cap(A^+)^j_{\alpha_2,\beta_2}\;.$$

Therefore, $A^{(1,1)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}=(A^-)^1_{\alpha_1,\beta 1}\cap(A^+)^1_{\alpha_2,\beta_2}$.

Example 1: Let A be an IVIFS on $\Phi(X)$, where

$$A=\{\langle x_1,[0.4,0.6],[0.2,0.4]\rangle,\langle x_2,[0.1,0.3],[0.3,0.6]\rangle,\langle x_3,[0.2,0.6],[0.1,0.4]\rangle,$$
$$\langle x_4,[0.5,0.7],[0.2,0.3]\rangle,\langle x_5,[0.3,0.5],[0.3,0.4]\rangle,\langle x_6,[0.1,0.6],[0.1,0.3]\rangle\}$$

and $\alpha=[0.2,0.4]$, $\beta=[0.2,0.3]\in[I]$.

Then

$$A^{(1,1)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}=\{\langle x_1,[0.4,0.6],[0.3,0.5]\rangle,\langle x_4,[0.5,0.7],[0.2,0.3]\rangle,\langle x_5,[0.3,0.5],[0.3,0.4]\rangle\}$$
$$(A^-)^1_{\alpha_1,\beta_1}=\{\langle x_1,[0.4,0.6],[0.3,0.5]\rangle,\langle x_4,[0.5,0.7],[0.2,0.3]\rangle,\langle x_5,[0.3,0.5],[0.3,0.4]\rangle\}$$
$$(A^+)^1_{\alpha_2,\beta_2}=\{\langle x_1,[0.4,0.6],[0.2,0.4]\rangle,\langle x_3,[0.2,0.6],[0.1,0.4]\rangle,$$
$$\langle x_4,[0.5,0.7],[0.2,0.3]\rangle,\langle x_5,[0.3,0.5],[0.3,0.4]\rangle,\langle x_6,[0.1,0.6],[0.1,0.3]\rangle\},$$

$$(A^-)^1_{\alpha_1,\beta_1}\cap(A^+)^1_{\alpha_2,\beta_2}=\{\langle x_1,[0.4,0.6],[0.3,0.5]\rangle,\langle x_4,[0.5,0.7],[0.2,0.3]\rangle,$$

$$\langle x_5,[0.3,0.5],\;[0.3,0.4]\rangle\}.$$

So, $A^{(1,1)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}=(A^-)^1_{\alpha_1,\beta_1}\cap(A^+)^1_{\alpha_2,\beta_2}$.

Proposition 2: Let $A\in\Phi(X)$ and $A^{(i,j)}_{\alpha,\beta}$ be the (i,j) th interval cut-set of IVIFS A where $\alpha=[\alpha_1,\alpha_2]$, $\beta=[\beta_1,\beta_2]$. Then,

$$A^{(2,2)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}\subseteq A^{(1,2)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}\subseteq A^{(1,1)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}$$

$$A^{(2,2)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}\subseteq A^{(2,1)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}\subseteq A^{(1,1)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}$$

$$A^{(4,4)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}\subseteq A^{(3,4)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}\subseteq A^{(3,3)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}$$

$$A^{(4,4)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}\subseteq A^{(4,3)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}\subseteq A^{(3,3)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}.$$

Proof : We prove

$$A^{(2,2)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}\subseteq A^{(1,2)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}\subseteq A^{(1,1)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}$$

and other proofs are similar.

Let, $x\in A^{(2,2)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}$

$$\Rightarrow\{x\in X\mid\{\langle x,[A^-(x)>\alpha_1,A^+(x)>\alpha_2],[B^-(x)>\beta_1,B^+(x)>\beta_2]\rangle\}$$

$$\subset\{x\in X\mid\{\langle x,[A^-(x)\geq\alpha_1,A^+(x)>\alpha_2],[B^-(x)\geq\beta_1,B^+(x)>\beta_2]\rangle\}$$
$$=A^{(1,2)}_{\alpha,\beta}$$

$$\subset\{x\in X\mid\{\langle x,[A^-(x)\geq\alpha_1,A^+(x)\geq\alpha_2],[B^-(x)\geq\beta_1,B^+(x)\geq\beta_2]\rangle\}$$
$$=A^{(1,1)}_{\alpha,\beta}.$$

So, $A^{(2,2)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}\subseteq A^{(1,2)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}\subseteq A^{(1,1)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}$.

Example 2: Let A be an IVIFS on $\Phi(X)$, where

$$A=\{\langle x_1,[0.4,0.6],[0.2,0.4]\rangle,\langle x_2,[0.1,0.3],[0.3,0.6]\rangle,\langle x_3,[0.2,0.6],[0.1,0.4]\rangle,$$
$$\langle x_4,[0.5,0.7],[0.2,0.3]\rangle,\langle x_5,[0.3,0.5],[0.3,0.4]\rangle,\langle x_6,[0.1,0.6],[0.1,0.3]\rangle\}$$

Let $\alpha=[0.2,0.4]$, $\beta=[0.2,0.3]\in[I]$.

We calculate $(2,2)$ th, $(1,2)$ th, $(1,1)$ th (α,β) interval cut-set of IVIFS A on $\Phi(X)$ that is,

$$A^{(2,2)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}=\langle x_5,[0.3,0.5],[0.3,0.4]\rangle$$

$$A^{(1,2)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]} = \{\langle x_1,[0.4,0.6],[0.3,0.5]\rangle,\langle x_5,[0.3,0.5],[0.3,0.4]\rangle\}$$

$$A^{(1,1)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]} = \{\langle x_1,[0.4,0.6],[0.3,0.5]\rangle,\langle x_4,[0.5,0.7],[0.2,0.3]\rangle,\langle x_5,[0.3,0.5],[0.3,0.4]\rangle\}$$

So, $A^{(2,2)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]} \subseteq A^{(1,2)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]} \subseteq A^{(1,1)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}$.

Proposition 3: *Let* $\alpha_1 = [\alpha_1^1, \alpha_1^2]$, $\alpha_2 = [\alpha_2^1, \alpha_2^2]$,

$\beta_1 = [\beta_1^1, \beta_1^2]$, $\beta_2 = [\beta_2^1, \beta_2^2] \in [I]$,

and $\alpha_1^1 \le \alpha_2^1$, $\alpha_1^2 \le \alpha_2^2$, $\beta_1^1 \le \beta_2^1$, $\beta_1^2 \le \beta_2^2$ then for IVIFS A on $\Phi(X)$ we have

(i) $A^{(1,1)}_{\alpha_2,\beta_2} \subseteq A^{(2,2)}_{\alpha_1,\beta_1}$

(ii) $A^{(3,3)}_{\alpha_2,\beta_2} \subseteq A^{(4,4)}_{\alpha_1,\beta_1}$.

Proof : (i) Let $x \in A^{(1,1)}_{[\alpha_2,\beta_2]}$

$$\Rightarrow \{x \in X : \langle x,[A^-(x) \ge \alpha_2^1, A^+(x) \ge \alpha_2^2],[B^-(x) \ge \beta_2^1, B^+(x) \ge \beta_2^2]\rangle\}$$

$$\subseteq \{x \in X : \langle x,[A^-(x) \ge \alpha_1^1, A^+(x) \ge \alpha_1^2],[B^-(x) \ge \beta_1^1, B^+(x) \ge \beta_1^2]\rangle\}$$

since, $\alpha_1^1 \le \alpha_2^1$, $\alpha_1^2 \le \alpha_2^2$, $\beta_1^1 \le \beta_2^1$, $\beta_1^2 \le \beta_2^2$.

$$\Rightarrow \{x \in X : \langle x,[A^-(x) > \alpha_1^1, A^+(x) > \alpha_1^2],[B^-(x) > \beta_1^1, B^+(x) > \beta_1^2]\rangle\}$$

$$\Rightarrow x \in A^{(2,2)}_{[\alpha_1 \beta_1]}.$$

Therefore, $A^{(1,1)}_{\alpha_2,\beta_2} \subseteq A^{(2,2)}_{\alpha_1,\beta_1}$.

(ii) The proof is to similar to (i).

Example 3: Let us consider an IVIFS A on $\Phi(X)$, where

$$A = \{\langle x_1,[0.4,0.6],[0.2,0.4]\rangle,\langle x_2,[0.1,0.3],[0.3,0.6]\rangle,\langle x_3,[0.2,0.6],[0.1,0.4]\rangle,$$

$$\langle x_4,[0.5,0.7],[0.2,0.3]\rangle,\langle x_5,[0.3,0.5],[0.3,0.4]\rangle,\langle x_6,[0.1,0.6],[0.1,0.3]\rangle\}.$$

Let $\alpha_1 = [0.2,0.4]$, $\beta_1 = [0.2,0.3]$, $\alpha_2 = [0.3,0.5]$,

$\beta_2 = [0.3,0.4] \in [I]$.

Such that $\alpha_1^1 \le \alpha_2^1$, $\alpha_1^2 \le \alpha_2^2$, $\beta_1^1 \le \beta_2^1$, $\beta_1^2 \le \beta_2^2$.

We calculate $(1,1)$ th (α_2, β_2) interval cut-set and $(2,2)$ th (α_1, β_1) interval cut-set of IVIFS of A on $\Phi(X)$. Then

$A^{(1,1)}_{\alpha_2,\beta_2} = \{\langle x_5,[0.3,0.5],[0.3,0.4]\rangle\}$ and

$A^{(2,2)}_{\alpha_1,\beta_1} = \{\langle x_5,[0.3,0.5],[0.3,0.4]\rangle\}$

So, $A^{(1,1)}_{\alpha_2,\beta_2} \subseteq A^{(2,2)}_{\alpha_1,\beta_1}$.

Definition 4: Let A be an IVIFS on $\Phi(X)$, and $\alpha = [\alpha_1, \alpha_2]$, $\beta = [\beta_1, \beta_2] \in [I]$, (i, j) th (α, β) interval cut-set of IVIFS on A is $A^{(i,j)}_{\alpha,\beta}$.

Then the complements of different interval cut-set on A are given below:

$$(A^{(1,1)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]})^c = \{\langle x,[A^-(x) \ngeq \alpha_1, A^+(x) \ngeq \alpha_2],[B^-(x) \ngeq \beta_1, B^+(x) \ngeq \beta_2]\rangle \mid x \in X\}$$

$$(A^{(1,2)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]})^c = \{\langle x,[A^-(x) \ngeq \alpha_1, A^+(x) \ngtr \alpha_2],[B^-(x) \ngeq \beta_1, B^+(x) \ngtr \beta_2]\rangle \mid x \in X\}$$

$$(A^{(2,1)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]})^c = \{\langle x,[A^-(x) \ngtr \alpha_1, A^+(x) \ngeq \alpha_2],[B^-(x) \ngeq \beta_1, B^+(x) \ngtr \beta_2]\rangle \mid x \in X\}$$

$$(A^{(2,2)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]})^c = \{\langle x,[A^-(x) \ngtr \alpha_1, A^+(x) \ngtr \alpha_2],[B^-(x) \ngtr \beta_1, B^+(x) \ngtr \beta_2]\rangle \mid x \in X\}.$$

From the definition of complements it is easy to observe the following results:

$$(i)(A^{(1,1)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]})^c = A^{(4,4)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}$$

$$(ii)(A^{(1,2)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]})^c = A^{(4,3)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}$$

$$(iii)(A^{(2,1)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]})^c = A^{(3,4)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}$$

$$(iv)(A^{(2,2)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]})^c = A^{(3,3)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}.$$

Definition 5: For $\alpha = [\alpha_1, \alpha_2]$, $\beta = [\beta_1, \beta_2] \in [I]$ and $A \in \Phi(X)$, we define two interval cartesian products, which convert each IVIFS to special type of IVIFS i.e. $(\alpha, \beta).A$, and $(\alpha, \beta) * A \in IVIFS$ defined for each $x \in X$ as follows:

$$(\alpha,\beta)A = \{x \in X : [\alpha_1 \wedge A^-(x), \alpha_2 \wedge A^+(x)],[\beta_1 \vee B^-(x), \beta_2 \vee B^+(x)]\}$$

$and (\alpha,\beta)^* A = \{x \in X : [\alpha_1 \vee A^-(x), \alpha_2 \vee A^+(x)][\beta_1 \wedge B^-(x), \beta_2 \wedge B^+(x)]\}$

where two fundamental operators $\wedge$ and $\vee$ are defined for all $x, y \in [0,1]$ such that

$(i)\ x \vee y = max(x, y)$ and $(ii)\ x \wedge y = min(x, y)$.

Proposition 4: *For* A, $B \in \Phi(X)$ *and* α, β, α_1, β_1, α_2, $\beta_2 \in [I]$ *then*

(a) For

$\alpha_1 \le \alpha_2, \beta_1 \le \beta_2, (i)\ (\alpha_1,\beta_1)A \subseteq (\alpha_2,\beta_2)A\ (ii)\ (\alpha_1,\beta_1)^* A \subseteq (\alpha_2,\beta_2)^* A$

(b) For

$A \subseteq B, (i)(\alpha,\beta).A \subseteq (\alpha,\beta).B\ (ii)(\alpha,\beta)^* A \subseteq (\alpha,\beta)^* B.$

Proof : The prove are straight forward.

Decomposition on interval cut-set of IVIFS

The principal role of interval cut-set of IVIFSs is their capability to represent IVIFSs. The representation of an arbitrary IVIFS A interms of interval cartesian product, which are defined interms of interval cut-set of IVIFS of A, is usually referred to as a decomposition theorem. We give three decomposition theorems of IVIFSs as follows:

Theorem 1: *Let* A *be an IVIFS on* $\Phi(X)$, *then for* $\alpha = [\alpha_1, \alpha_2]$, $\beta = [\beta_1, \beta_2] \in [I]$

$$A = \bigcup_{\alpha,\beta \in [I]} (\alpha,\beta).A^{(1,j)}_{\alpha,\beta},\ j = 1,2$$

Proof : To prove this theorem we assume that the set A has n elements of the form

$A = \{\langle x_i, a_i, b_i \rangle\}$ where $i = 1,2,\cdots n$ and $a_i = [a_{1_i}, a_{2_i}]$, $b = [b_{1_i}, b_{2_i}]$, $\alpha = [\alpha_1, \alpha_2]$, $\beta = [\beta_1, \beta_2] \in [I]$

$$Now \bigcup_{\alpha,\beta \in [I]} (\alpha,\beta).A^{(1,j)}_{\alpha,\beta}$$

$$= \{\langle x_i, [\max_{\alpha_1}\{\alpha_1 \wedge a_{1_i}\}, \max_{\alpha_2}\{\alpha_2 \wedge a_{2_i}\}][\min_{\beta_1}\{\beta_1 \vee b_{1_i}\}, \min_{\beta_2}\{\beta_2 \vee b_{2_i}\}]\rangle\}$$

Here two cases may arise. Case-(i): $a_i < \alpha$, $b_i < \beta$ and case-(ii): $a_i \ge \alpha$, $b_i \ge \beta$

Case-(i): When $a < \alpha$, $b < \beta$ then

$A^{(1,j)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]} = \phi$ and therefore

$$([\alpha_1,\alpha_2],[\beta_1,\beta_2]).A^{(1,j)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]} = \phi.$$

Case-(ii): When $a_i \ge \alpha$, $b_i \ge \beta$ then we have

$$A^{(1,j)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]} = \{\langle x_i, [a_{1_i}, a_{2_i}], [b_{1_i}, b_{2_i}]\rangle\}$$

$$([\alpha_1,\alpha_2],[\beta_1,\beta_2]).A^{(1,j)}_{[\alpha_1,\alpha_2],[\beta_1,\beta_2]}$$

$$= \{\langle x, [\max_{\alpha_1}\{\alpha_1 \wedge a_{1_i}\}, \max_{\alpha_2}\{\alpha_2 \wedge a_{2_i}\}][\min_{\beta_1}\{\beta_1 \vee b_{1_i}\}, \min_{\beta_2}\{\beta_2 \vee b_{2_i}\}]\rangle\}$$

$$= \{\langle x_i, [a_{1_i}, a_{2_i}], [b_{1_i}, b_{2_i}]\rangle\}.$$

From Case $-(i)$ *and Case* $-(ii)$ *we have*

$$\bigcup_{\alpha,\beta \in [I]} (\alpha,\beta).A^{(1,j)}_{\alpha,\beta}$$

$$= \bigcup_{\substack{\alpha < a_i \\ \beta < b_i}} (\alpha,\beta).A^{(1,j)}_{\alpha,\beta} \bigcup_{\substack{\alpha \ge a_i \\ \beta \ge b_i}} (\alpha,\beta).A^{(1,j)}_{\alpha,\beta}$$

$$= \{\langle x_i, [a_{1_i}, a_{2_i}], [b_{1_i}, b_{2_i}]\rangle\}$$
$$= A.$$

Therefore for every $x_i \in X$.

$$A = \bigcup_{\alpha,\beta \in [I]} (\alpha,\beta).A^{(1,j)}_{\alpha,\beta},\ j = 1,2.$$

Example 4: Let us consider an IVIFS A on $\Phi(X)$, where

$$A = \{\langle x_1, [0.4,0.6],[0.3,0.4]\rangle, \langle x_2, [0.1,0.3],[0.3,0.6]\rangle, \langle x_3, [0.2,0.6],[0.1,0.4]\rangle,$$

$$\langle x_4, [0.5,0.7],[0.2,0.3]\rangle, \langle x_5, [0.3,0.5],[0.3,0.4]\rangle, \langle x_6, [0.1,0.6],[0.1,0.3]\rangle\}$$

Now we calculate following interval cut-sets of IVIFS A

$$A^{(1,1)}_{[0.4,0.6],[0.3,0.4]} = \{\langle x_1, [0.4,0.6],[0.3,0.4]\rangle\}$$

$$A^{(1,1)}_{[0.1,0.3],[0.3,0.6]} = \{\langle x_2,[0.1,0.3],[0.3,0.6]\rangle\}$$

$$A^{(1,1)}_{[0.2,0.6],[0.1,0.4]} = \{\langle x_1,[0.4,0.6],[0.3,0.4]\rangle,\langle x_3,[0.2,0.6],[0.1,0.4]\rangle\}$$

$$A^{(1,1)}_{[0.5,0.7],[0.2,0.3]} = \{\langle x_4,[0.5,0.7],[0.2,0.3]\rangle\}$$

$$A^{(1,1)}_{[0.3,0.5],[0.3,0.4]} = \{\langle x_5,[0.3,0.5],[0.3,0.4]\rangle\}$$

$$A^{(1,1)}_{[0.1,0.6],[0.1,0.3]} = \{\langle x_1,[0.4,0.6],[0.3,0.4]\rangle,\langle x_3,[0.2,0.6],[0.1,0.4]\rangle,$$
$$\langle x_4,[0.5,0.7],[0.2,0.3]\rangle,\langle x_6,[0.1,0.6],[0.1,0.3]\rangle\}$$

we calculate interval cartesian product as

$$([0.4,0.6],[0.3,0.4]).A^{(1,1)}_{[0.4,0.6],[0.3,0.4]} = \{\langle x_1,[0.4,0.6],[0.3,0.4]\rangle\}$$

$$([0.1,0.3],[0.3,0.6]).A^{(1,1)}_{[0.1,0.3],[0.3,0.6]} = \{\langle x_2,[0.1,0.3],[0.3,0.6]\rangle\}$$

$$([0.2,0.6],[0.1,0.4]).A^{(1,1)}_{[0.2,0.6],[0.1,0.4]} = \{\langle x_1,[0.2,0.6],[0.2,0.4]\rangle,\langle x_3,[0.2,0.6],[0.1,0.4]\rangle\}$$

$$([0.5,0.7],[0.2,0.3]).A^{(1,1)}_{[0.5,0.7],[0.2,0.3]} = \{\langle x_4,[0.5,0.7],[0.2,0.3]\rangle\}$$

$$([0.3,0.5],[0.3,0.4]).A^{(1,1)}_{[0.3,0.5],[0.3,0.4]} = \{\langle x_5,[0.3,0.5],[0.3,0.4]\rangle\}$$

$$([0.1,0.6],[0.1,0.3]).A^{(1,1)}_{[0.1,0.6],[0.1,0.3]} = \{\langle x_1,[0.1,0.6],[0.2,0.4]\rangle,\langle x_3,[0.1,0.6],[0.1,0.4]\rangle,$$
$$\langle x_4,[0.1,0.6],[0.2,0.3]\rangle,\langle x_6,[0.1,0.6],[0.1,0.3]\rangle\},$$

$$\bigcup_{\alpha,\beta\in[I]}(\alpha,\beta).A^{(1,j)}_{\alpha,\beta}$$

$$=\{\langle x_1,[\max\{0.4,0.1\},\max\{0.6,0.6\}],[\min\{0.2,0.2\},\min\{0.4,0.4\}]\rangle\},$$

$$\{\langle x_2,[\max\{0.1\},\max\{0.3\}],[\min\{0.3\},\min\{0.6\}]\rangle\},$$
$$\{\langle x_3,[\max\{0.2,0.1\},\max\{0.6,0.6\}],[\min\{0.1,0.1\},\min\{0.4,0.4\}]\rangle\},$$

$$\{\langle x_4,[\max\{0.5,0.1\},\max\{0.7,0.6\}],[\min\{0.2,0.2\},\min\{0.3,0.3\}]\rangle\},$$
$$\{\langle x_5,[\max\{0.3\},\max\{0.5\}][\min\{0.3\},\min\{0.4\}]\rangle\},$$

$$\{\langle x_6,[\max\{0.1\}\max\{0.6\}],[\min\{0.1\},\min\{0.3\}]\rangle\},$$

$$=\{\langle x_1,[0.4,0.6],[0.3,0.4]\rangle,\langle x_2,[0.1,0.3],[0.3,0.6]\rangle,\langle x_3,[0.2,0.6],[0.1,0.4]\rangle,$$
$$\langle x_4,[0.5,0.7],[0.2,0.3]\rangle,\langle x_5,[0.3,0.5],[0.3,0.4]\rangle,\langle x_6,[0.1,0.6],[0.1,0.3]\rangle\}$$
$$= A.$$

Hence the result.

Theorem 2: *Let* A *be an IVIFS on* $\Phi(X)$*, then for* $\alpha=[\alpha_1,\alpha_2]$, $\beta=[\beta_1,\beta_2]\in[I]$

$$A = (\bigcap_{\alpha,\beta\in[I]}(\alpha,\beta)*A^{(4,i)}_{\alpha,\beta})^c, i=3,4.$$

Suppose H be a mapping from $[I]$ to $P(X)$ where $P(X)$ is the crisp set, $H:[I]\to P(X)$, for every $\alpha,\beta\in[I]$. We have $H(\alpha,\beta)\in P(X)$. Obviously interval cut-set of interval-valued intuitionistic fuzzy set, $A_{\alpha,\beta}\in P(X)$, it means that mapping H indeed exits. Based on Theorems 1 and 2, we have the following theorem in general. For $A\in IVIFS$, we have

Theorem 3: *(1) If* $A^{(1,2)}_{\alpha,\beta}\subseteq H(\alpha,\beta)\subseteq A^{(1,1)}_{\alpha,\beta}$ *then*

$$A=\bigcup_{\alpha,\beta\in[I]}(\alpha,\beta).H(\alpha,\beta)$$

and if $A^{(4,4)}_{\alpha,\beta}\subseteq H(\alpha,\beta)\subseteq A^{(4,3)}_{\alpha,\beta}$ then

$$A=(\bigcup_{\alpha,\beta\in[I]}(\alpha,\beta)*H(\alpha,\beta)^c)^c.$$

Proof. The proofs are straightforward.

In the following we investigate the relation of interval cut-set of IVIFS A and mapping H. For simplicity, we denote $\alpha_1=[\alpha_1^1,\alpha_1^2]$, $\alpha_2=[\alpha_2^1,\alpha_2^2]$, $\beta_1=[\beta_1^1,\beta_1^2]$, $\beta_2=[\alpha_2^1,\beta_2^2]$ and α_1, α_2, β_1, $\beta_2\in[I]$, then we give some properties of interval cut-set of IVIFS and mapping H.

Proposition 5: If relation $A^{(2,2)}_{\alpha,\beta}\subseteq H(\alpha,\beta)\subseteq A^{(1,1)}_{\alpha,\beta}$ is satisfy for $A^{(2,2)}_{\alpha,\beta}$, $A^{(1,1)}_{\alpha,\beta}$ and $H(\alpha,\beta)$ then

$$(a)\ \alpha_1,\alpha_2,\beta_1,\beta_2\in[I], where\ \alpha_1=[\alpha_1^1,\alpha_1^2],\alpha_2=[\alpha_2^1,\alpha_2^2],\beta_1=[\beta_1^1,\beta_1^2],\beta_2=[\beta_2^1,\beta_2^2]$$

$$and\ \alpha_1^i<\alpha_2^i,\beta_1^i<\beta_2^i, i=1,2\Rightarrow H(\alpha_1,\beta_1)\supseteq H(\alpha_2,\beta_2),$$
$$(b)\ A^{(1,1)}_{\alpha,\beta}=\bigcap_{\gamma<\alpha}\bigcap_{\delta<\beta}H(\gamma,\delta), where\ \gamma=[\gamma_1,\gamma_2],\delta=[\delta_1,\delta_2]\in[I]\ and\ \alpha_1\neq 0,\beta_1\neq 0.$$

$$(c)\ A^{(2,2)}_{\alpha,\beta}=\bigcup_{\gamma>\alpha}\bigcup_{\delta>\beta}H(\gamma,\delta), where\ \gamma=[\gamma_1,\gamma_2],\delta=[\delta_1,\delta_2]\in[I]\ and\ \alpha_2\neq 1,\beta_2\neq 1.$$

The proof of the results of the above proposition is simple. Here we verify the result by considering an example.

Example 5: Let us consider an IVIFS A on $\Phi(X)$, where

$$A=\{\langle x_1,[0.4,0.6],[0.3,0.4]\rangle,\langle x_2,[0.3,0.7],[0.2,0.3]\rangle,\langle x_3,[0.2,0.5],[0.3,0.5]\rangle,$$
$$\langle x_4,[0.2,0.5],[0.1,0.2]\rangle,\langle x_5,[0.5,0.6],[0.3,0.4]\rangle,\langle x_6,[0.4,0.6],[0.2,0.4]\rangle$$
$$\langle x_7,[0.3,0.5],[0.4,0.5]\rangle,\langle x_8,[0.3,0.5],[0.2,0.3]\rangle\},$$

(a) Let us consider, $\alpha_1=[0.3,0.5]$, $\beta_1=[0.2,0.3]$ and

$\alpha_2=[0.4,0.6$, $\beta_2=[0.3,0.4]\in[I]$ such that

$\alpha_1<\alpha_2,\beta_1<\beta_2$, then we have,

$$A^{(1,1)}_{[0.3,0.5],[0.2,0.3]}=\{\langle x_1,[0.4,0.6],[0.3,0.4]\rangle,\langle x_2,[0.3,0.7],[0.2,0.3]\rangle,$$
$$\langle x_5,[0.5,0.6],[0.3,0.4]\rangle,\langle x_6,[0.4,0.6],[0.2,0.4]\rangle,\langle x_7,[0.3,0.5],[0.4,0.5]\rangle,\langle x_8,[0.3,0.5],[0.2,0.3]\rangle\}$$
$$A^{(2,2)}_{[0.3,0.5],[0.2,0.3]}=\{\langle x_1,[0.4,0.6],[0.3,0.4]\rangle,\langle x_5,[0.5,0.6],[0.3,0.4]\rangle,\langle x_7,[0.3,0.5],[0.4,0.5]\rangle\}$$

Therefore,

$$H([0.3,0.5],[0.2,0.3])=\{\langle x_1,[0.4,0.6],[0.3,0.4]\rangle,\langle x_5,[0.5,0.6],[0.3,0.4]\rangle,\langle x_7,[0.3,0.5],[0.4,0.5]\rangle\}$$

Again we calculate

$$A^{(1,1)}_{[0.4,0.6],[0.3,0.4]}=\{\langle x_1,[0.4,0.6],[0.3,0.4]\rangle,\langle x_5,[0.5,0.6],[0.3,0.4]\rangle\},$$

And, $A^{(2,2)}_{[0.4,0.6],[0.3,0.4]}=\{\phi\}$

Therefore,

$$H([0.4,0.6],[0.3,0.4])=\{\langle x_1,[0.4,0.6],[0.3,0.4]\rangle,\langle x_5,[0.5,0.6],[0.3,0.4]\rangle\}.$$

Hence,

$$H([0.3,0.5],[0.2,0.3])\supseteq H([0.4,0.6],[0.3,0.4]).$$

(b) Let us consider,

$\alpha=[0.4,0.6],\beta=[0.3,0.5],\gamma,\delta\in[I]$,

such that $\gamma<\alpha$, $\delta<\beta$, then

$$A^{(1,1)}_{[0.4,0.6],[0.3,0.4]}=\bigcap_{\gamma<\alpha}\bigcap_{\delta<\beta}H(\gamma,\delta).$$

$$=\bigcap_{\gamma<\alpha}H([0.4,0.6],[0.2,0.3])\bigcap_{\gamma<\alpha}H([0.4,0.6],[0.1,0.2])$$

$$=\bigcap H([0.2,0.5],[0.2,0.3])\bigcap H([0.2,0.5],[0.1,0.2])\bigcap H([0.3,0.5],[0.2,0.3]$$
$$\bigcap H([0.3,0.5],[0.1,0.2])$$

Now we calculate

$$H([0.2,0.5],[0.2,0.3])=\{\langle x_1,[0.4,0.6],[0.3,0.4]\rangle,\langle x_5,[0.5,0.6],[0.3,0.4]\rangle\}$$

$$H([0.3,0.5],[0.2,0.3])=\{\langle x_1,[0.4,0.6],[0.3,0.4]\rangle,\langle x_5,[0.5,0.6],[0.3,0.4]\rangle\}$$
$$H([0.3,0.5],[0.1,0.2])=\{\langle x_1,[0.4,0.6],[0.3,0.4]\rangle,\langle x_5,[0.5,0.6],[0.3,0.4]\rangle,\langle x_6,[0.4,0.6],[0.2,0.4]\rangle\}$$

Therefore,

$$H([0.2,0.5],[0.2,0.3])\bigcap H([0.2,0.5],[0.1,0.2])\bigcap H([0.3,0.5],[0.2,0.3])\bigcap H([0.3,0.5],[0.1,0.$$
$$=\{\langle x_1,[0.4,0.6],[0.3,0.4]\rangle,\langle x_5,[0.5,0.6],[0.3,0.4]\rangle\}$$

$$\bigcap\{\langle x_1,[0.4,0.6],[0.3,0.4]\rangle,\langle x_5,[0.5,0.6],[0.3,0.4]\rangle,\langle x_7,[0.3,0.5],[0.4,0.5]\rangle$$
$$\bigcap\{\langle x_1,[0.4,0.6],[0.3,0.4]\rangle,\langle x_5,[0.5,0.6],[0.3,0.4]\rangle\}$$

$$\bigcap\{\langle x_1,[0.4,0.6],[0.3,0.4]\rangle,\langle x_5,[0.5,0.6],[0.3,0.4]\rangle,\langle x_6,[0.4,0.6],[0.2,0.4]\rangle\},$$
$$=\{\langle x_1,[0.4,0.6],[0.3,0.4]\rangle,\langle x_5,[0.5,0.6],[0.3,0.4]\rangle\}$$

$$=A^{(1,1)}_{[0.4,0.6],[0.3,0.4]}$$

$$Hence,\ A^{(1,1)}_{[0.4,0.6],[0.3,0.4]}=\bigcap_{\gamma<\alpha}\bigcap_{\delta<\beta}H(\gamma,\delta).$$

(c) Let us consider,

$\alpha=[0.3,0.5],\beta=[0.2,0.3],\gamma,\delta\in[I]$, such that

$\gamma>\alpha,\delta>\beta$, then

$$A^{(2,2)}_{[0.3,0.5],[0.2,0.3]}=\bigcup_{\gamma>\alpha}\bigcup_{\delta>\beta}H(\gamma,\delta).$$

$$=\bigcup_{\gamma>\alpha}H([0.3,0.5],[0.3,0.4])$$
$$=H([0.4,0.6],[0.3,0.4])\bigcup H([0.5,0.6],[0.3,0.4])$$

$$Now,\ H([0.4,0.6],[0.3,0.4])=\{\langle x_1,[0.4,0.6],[0.3,0.4]\rangle,\langle x_5,[0.5,0.6],[0.3,0.4]\rangle\}$$

$$H([0.4,0.6],[0.3,0.4])=\{\langle x_5,[0.5,0.6],[0.3,0.4]\rangle\},$$

$$H([0.4,0.6],[0.3,0.4])\bigcup H([0.4,0.6],[0.3,0.4])$$

$$=\{\langle x_1,[0.4,0.6],[0.3,0.4]\rangle,\langle x_5,[0.5,0.6],[0.3,0.4]\rangle\}\bigcup\{\langle x_5,[0.5,0.6],[0.3,0.4]\rangle\}$$
$$=\{\langle x_1,[0.4,0.6],[0.3,0.4]\rangle,\langle x_5,[0.5,0.6],[0.3,0.4]\rangle\}$$

$$Hence,\ A^{(2,2)}_{[0.3,0.5][0.2,0.3]} = \bigcup_{\gamma>\alpha}\bigcup_{\delta>\beta} H(\gamma,\delta).$$

Proposition 6: *If $A^{(2,2)}_{\alpha,\beta}$, $A^{(1,2)}_{\alpha,\beta}$ and $H(\alpha,\beta)$ satisfy the condition $A^{(2,2)}_{\alpha,\beta} \subseteq H(\alpha,\beta) \subseteq A^{(1,2)}_{\alpha,\beta}$ then*

$$(a)\alpha_1,\alpha_2,\beta_1,\beta_2 \in [I],\ where\ \alpha_1 = [\alpha_1^1,\alpha_1^2],$$

$$\alpha_2 = [\alpha_2^1,\alpha_2^2],\ \beta_1 = [\beta_1^1,\beta_1^2],\ \beta_2 = [\beta_2^1,\beta_2^2]$$

$$and\ \alpha_1^i < \alpha_2^i, \beta_1^i < \beta_2^i, i=1,2 \Rightarrow H(\alpha_1,\beta_1) \supseteq H(\alpha_2,\beta_2).$$

$$(b)A^{(1,2)}_{\alpha,\beta} = \bigcap_{\substack{\gamma_1<\alpha_1\\ \delta_1<\beta_1}} H([\gamma_1,\alpha_2],[\delta_1,\beta_2]), \gamma_1,\delta_1 \in [0,1], \alpha_1 \neq 0, \beta_1 \neq 0.$$

$$(c)A^{(2,2)}_{\alpha,\beta} = \bigcup_{\substack{\gamma_1>\alpha_1\\ \delta_1>\beta_1}} H([\gamma_1,\alpha_2],[\delta_1,\beta_2]), \gamma_1,\delta_1 \in [0,1], \alpha_1 \neq 1, \beta_1 \neq 1.$$

Proposition 7: If the relation $A^{(2,2)}_{\alpha,\beta} \subseteq H(\alpha,\beta) \subseteq A^{(2,1)}_{\alpha,\beta}$ is

true for $A^{(2,2)}_{\alpha,\beta}$, $A^{(2,1)}_{\alpha,\beta}$ and $H(\alpha,\beta)$ *then*

$$(a)\alpha_1,\alpha_2,\beta_1,\beta_2 \in [I], where\ \alpha_1 = [\alpha_1^1,\alpha_1^2], \alpha_2 = [\alpha_2^1,\alpha_2^2], \beta_1 = [\beta_1^1,\beta_1^2], \beta_2 = [\beta_2^1,\beta_2^2]$$

$$and\ \alpha_1^i < \alpha_2^i, \beta_1^i < \beta_2^i, i=1,2 \Rightarrow H(\alpha_1,\beta_1) \supseteq H(\alpha_2,\beta_2).$$

$$(b)A^{(2,1)}_{\alpha,\beta} = \bigcap_{\substack{\gamma_2<\alpha_2\\ \delta_2<\beta_2}} H([\alpha_1,\gamma_2],[\beta_1,\delta_2]), \gamma_2,\delta_2 \in [0,1], \alpha_2 \neq 0, \beta_2 \neq 0.$$

$$(c)A^{(2,2)}_{\alpha,\beta} = \bigcup_{\substack{\gamma_2>\alpha_2\\ \delta_2>\beta_2}} H([\alpha_1,\gamma_2],[\beta_1,\delta_2]), \gamma_2,\delta_2 \in [0,1], \alpha_2 \neq 1, \beta_2 \neq 1.$$

Proposition 8: If the condition $A^{(2,1)}_{\alpha,\beta} \subseteq H(\alpha,\beta) \subseteq A^{(1,1)}_{\alpha,\beta}$ holds for $A^{(2,1)}_{\alpha,\beta}$, $A^{(1,1)}_{\alpha,\beta}$ and $H(\alpha,\beta)$ then

$$(a)\alpha_1,\alpha_2,\beta_1,\beta_2 \in [I], where\ \alpha_1 = [\alpha_1^1,\alpha_1^2], \alpha_2 = [\alpha_2^1,\alpha_2^2], \beta_1 = [\beta_1^1,\beta_1^2], \beta_2 = [\beta_2^1,\beta_2^2]$$

$$and\ \alpha_1^i < \alpha_2^i, \beta_1^i < \beta_2^i, i=1,2 \Rightarrow H(\alpha_1,\beta_1) \supseteq H(\alpha_2,\beta_2).$$

$$(b)A^{(1,1)}_{\alpha,\beta} = \bigcap_{\substack{\gamma_1<\alpha_1\\ \delta_1<\beta_1}} H([\gamma_1,\alpha_2],[\delta_1,\beta_2]), \gamma_1,\delta_1 \in [0,1], \alpha_1 \neq 0, \beta_1 \neq 0.$$

$$(c)A^{(2,1)}_{\alpha,\beta} = \bigcup_{\substack{\gamma_1>\alpha_1\\ \delta_1>\beta_1}} H([\gamma_1,\alpha_2],[\delta_1,\beta_2]), \gamma_1,\delta_1 \in [0,1], \alpha_1 \neq 1, \beta_1 \neq 1.$$

Proposition 9: *If $A^{(1,2)}_{\alpha,\beta}$, $A^{(1,1)}_{\alpha,\beta}$ and $H(\alpha,\beta)$ satisfy*

$A^{(1,2)}_{\alpha,\beta} \subseteq H(\alpha,\beta) \subseteq A^{(1,1)}_{\alpha,\beta}$ *then*

$$(a)\alpha_1,\alpha_2,\beta_1,\beta_2 \in [I], where\ \alpha_1 = [\alpha_1^1,\alpha_1^2], \alpha_2 = [\alpha_2^1,\alpha_2^2], \beta_1 = [\beta_1^1,\beta_1^2], \beta_2 = [\beta_2^1,\beta_2^2]$$

$$and\ \alpha_1^i < \alpha_2^i, \beta_1^i < \beta_2^i, i=1,2 \Rightarrow H(\alpha_1,\beta_1) \supseteq H(\alpha_2,\beta_2).$$

$$(b)A^{(1,1)}_{\alpha,\beta} = \bigcap_{\substack{\gamma_2<\alpha_2\\ \delta_2<\beta_2}} H([\alpha_1,\gamma_2],[\beta_1,\delta_2]), \gamma_1,\delta_1 \in [0,1], \alpha_2 \neq 0, \beta_2 \neq 0.$$

$$(c)A^{(1,2)}_{\alpha,\beta} = \bigcup_{\substack{\gamma_2>\alpha_2\\ \delta_2>\beta_2}} H([\alpha_1,\gamma_2],[\beta_1,\delta_2]), \gamma_1,\delta_1 \in [0,1], \alpha_2 \neq 1, \beta_2 \neq 1.$$

Proposition 10: If the relation $A^{(4,4)}_{\alpha,\beta} \subseteq H(\alpha,\beta) \subseteq A^{(3,3)}_{\alpha,\beta}$ is true for $A^{(4,4)}_{\alpha,\beta}$, $A^{(3,3)}_{\alpha,\beta}$ and $H(\alpha,\beta)$ then

$$(a)\alpha_1,\alpha_2,\beta_1,\beta_2 \in [I], where\ \alpha_1 = [\alpha_1^1,\alpha_1^2], \alpha_2 = [\alpha_2^1,\alpha_2^2], \beta_1 = [\beta_1^1,\beta_1^2], \beta_2 = [\beta_2^1,\beta_2^2]$$

$$and\ \alpha_1^i < \alpha_2^i, \beta_1^i < \beta_2^i, i=1,2 \Rightarrow H([\alpha_1,\beta_1]) \subseteq H(\alpha_2,\beta_2).$$

$$(b)A^{(4,4)}_{\alpha,\beta} = \bigcup_{\gamma<\alpha}\bigcup_{\delta<\beta} H(\gamma,\delta), where\ \gamma = [\gamma_1,\gamma_2], \delta = [\delta_1,\delta_2] \in [I]\ and\ \alpha_1 \neq 0, \beta_1 \neq 0.$$

$$(c)A^{(3,3)}_{\alpha,\beta} = \bigcap_{\gamma>\alpha}\bigcap_{\delta>\beta} H([\gamma,\delta]), where\ \gamma = [\gamma_1,\gamma_2]\ \delta = [\delta_1,\delta_2] \in [I]\ and\ \alpha_2 \neq 1, \beta_2 \neq 1$$

Proposition 11: If $A^{(4,4)}_{\alpha,\beta}$, $A^{(4,3)}_{\alpha,\beta}$ and $H(\alpha,\beta)$ satisfy the condition $A^{(4,4)}_{\alpha,\beta} \subseteq H(\alpha,\beta) \subseteq A^{(4,3)}_{\alpha,\beta}$ then

$$(a)\alpha_1,\alpha_2,\beta_1,\beta_2 \in [I], where\ \alpha_1 = [\alpha_1^1,\alpha_1^2], \alpha_2 = [\alpha_2^1,\alpha_2^2], \beta_1 = [\beta_1^1,\beta_1^2], \beta_2 = [\beta_2^1,\beta_2^2]$$

$$and\ \alpha_1^i < \alpha_2^i, \beta_1^i < \beta_2^i, i=1,2 \Rightarrow H(\alpha_1,\beta_1) \subseteq H(\alpha_2,\beta_2).$$

$$(b)A^{(4,4)}_{\alpha,\beta} = \bigcup_{\substack{\gamma_2<\alpha_2\\ \delta_2<\beta_2}} H([\alpha_1,\gamma_2],[\beta_1,\delta_2]), \gamma_2,\delta_2 \in [0,1], \alpha_2 \neq 0, \beta_2 \neq 0.$$

$$(c)A^{(4,3)}_{\alpha,\beta} = \bigcap_{\substack{\gamma_2>\alpha_2\\ \delta_2>\beta_2}} H([\alpha_1,\gamma_2],[\beta_1,\delta_2]), \gamma_2,\delta_2 \in [0,1], \alpha_2 \neq 1, \beta_2 \neq 1.$$

Proposition 12: If the condition $A_{\alpha,\beta}^{(4,4)} \subseteq H(\alpha,\beta) \subseteq A_{\alpha,\beta}^{(3,4)}$ is true for $A_{\alpha,\beta}^{(4,4)}$, $A_{\alpha,\beta}^{(3,4)}$ and $H(\alpha,\beta)$ then

$$(a)\alpha_1,\alpha_2,\beta_1,\beta_2 \in [I], where\ \alpha_1=[\alpha_1^1,\alpha_1^2], \alpha_2=[\alpha_2^1,\alpha_2^2], \beta_1=[\beta_1^1,\beta_1^2], \beta_2=[\beta_2^1,\beta_2^2]$$

$$and\ \alpha_1^i < \alpha_2^i, \beta_1^i < \beta_2^i, i=1,2 \Rightarrow H(\alpha_1,\beta_1) \subseteq H(\alpha_2,\beta_2).$$

$$(b)A_{\alpha,\beta}^{(4,4)} = \bigcup_{\substack{\gamma_1<\alpha_1\\ \delta_1<\beta_1}} H([\gamma_1,\alpha_2],[\delta_1,\beta_2]), \gamma_1,\delta_1 \in [0,1] \alpha_1 \neq 0, \beta_1 \neq 0.$$

$$(c)A_{\alpha,\beta}^{(3,3)} = \bigcap_{\substack{\gamma_1>\alpha_1\\ \delta_1>\beta_1}} H([\gamma_1,\alpha_2],[\delta_1,\beta_2]), \gamma_1,\delta_1 \in [0,1] \alpha_1 \neq 1, \beta_1 \neq 1.$$

Proposition 13: *If* $A_{\alpha,\beta}^{(3,4)}$, $A_{\alpha,\beta}^{(3,3)}$ *and* $H(\alpha,\beta)$ *satisfy the relation* $A_{\alpha,\beta}^{(3,4)} \subseteq H(\alpha,\beta) \subseteq A_{\alpha,\beta}^{(3,3)}$ *then*

$$(a)\alpha_1,\alpha_2,\beta_1,\beta_2 \in [I], where\ \alpha_1=[\alpha_1^1,\alpha_1^2], \alpha_2=[\alpha_2^1,\alpha_2^2], \beta_1=[\beta_1^1,\beta_1^2], \beta_2=[\beta_2^1,\beta_2^2]$$

$$and\ \alpha_1^i < \alpha_2^i, \beta_1^i < \beta_2^i, i=1,2 \Rightarrow H(\alpha_1,\beta_1) \subseteq H(\alpha_2,\beta_2).$$

$$(b)A_{\alpha,\beta}^{(3,3)} = \bigcup_{\substack{\gamma_2<\alpha_2\\ \delta_2<\beta_2}} H([\alpha_1,\gamma_2],[\beta_1,\delta_2]), \gamma_2,\delta_2 \in [0,1] \alpha_2 \neq 0, \beta_2 \neq 0.$$

$$(c)A_{\alpha,\beta}^{(3,4)} = \bigcap_{\substack{\gamma_2>\alpha_2\\ \delta_2>\beta_2}} H([\alpha_1,\gamma_2],[\beta_1,\delta_2]), \gamma_2,\delta_2 \in [0,1], \alpha_2 \neq 1, \beta_2 \neq 1.$$

Proposition 14: *If the relation* $A_{\alpha,\beta}^{(4,3)} \subseteq H(\alpha,\beta) \subseteq A_{\alpha,\beta}^{(3,3)}$ *holds for* $A_{\alpha,\beta}^{(4,3)}$, $A_{\alpha,\beta}^{(3,3)}$ *and* $H(\alpha,\beta)$ *then*

$$(a)\alpha_1,\alpha_2,\beta_1,\beta_2 \in [I], where\ \alpha_1=[\alpha_1^1,\alpha_1^2], \alpha_2=[\alpha_2^1,\alpha_2^2], \beta_1=[\beta_1^1,\beta_1^2], \beta_2=[\beta_2^1,\beta_2^2]$$

$$and\ \alpha_1^i < \alpha_2^i, \beta_1^i < \beta_2^i, i=1,2 \Rightarrow H(\alpha_1,\beta_1) \subseteq H(\alpha_2,\beta_2).$$

$$(b)A_{\alpha,\beta}^{(4,3)} = \bigcup_{\substack{\gamma_1<\alpha_1\\ \delta_1<\beta_1}} H([\gamma_1,\alpha_2],[\delta_1,\beta_2]), \gamma_1,\delta_1 \in [0,1] \alpha_1 \neq 0, \beta_1 \neq 0.$$

$$(c)A_{\alpha,\beta}^{(3,3)} = \bigcap_{\substack{\gamma_1>\alpha_1\\ \delta_1>\beta_1}} H([\gamma_1,\alpha_2],[\delta_1,\beta_2]), \gamma_1,\delta_1 \in [0,1], \alpha_1 \neq 1, \beta_1 \neq 1.$$

CONCLUSION

We introduce the different types of interval cut-sets of IVIFSs and complements. Also we proved some results of those interval cut-sets of IVIFS with some examples. Also we present three decomposition theorems of IVIFS and a mapping H with some properties. These works can be used in setting up the basic theory of interval-valued fuzzy set. In the next paper, we try to make different types of interval cut-sets of GIVIFS and related properties of interval cut-sets of GIVIFS.

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