

Full Length Research Paper

A characterization of the projective transformation

Yasemin Alagöz* and Ziya Soyuçok

Yildiz Technical University, Department of Mathematics, Davutpasa, 34210, Esenler, Istanbul, Turkey.

Accepted 4 December, 2009

We consider the transformations preserving the asymptotic directions of hypersurfaces in n-dimensional Euclidean space and we obtain the following characterization of the projective transformation: In n-dimensional Euclidean space, a transformation preserves the asymptotic directions of hypersurfaces if and only if it is the projective transformation.

Key words: Projective transformation, hypersurfaces, asymptotic directions.

INTRODUCTION

In the Euclidean space, the projective transformation has some interesting properties. For example, it has the following characterization: A transformation is the projective one if and only if it transforms a straight line to the other straight line (Lovett, 1895/1896). In the 3-dimensional Euclidean space, the projective transformation transforms an infinitesimally rigid surface to the other infinitesimally rigid surface, that is, it preserves the infinitesimal rigidity (Efimov, 1951; Özkan and Soyuçok, 1987). The projective transformation also preserves the asymptotic lines of a surface (Eisenhart, 1960). In Uras, (1995), the inverse of that problem is considered and it is obtained that the most transformation preserving the asymptotic lines of surfaces in 3-dimensional Euclidean space is the projective one. But that paper has very long calculations and it seems very difficult to generalize for the n-dimensional space by using given method. Moreover, since it has some errors that transformation is not the general projective transformation (Alagöz Y, Soyuçok Z, Bulletin of the Technical University of Istanbul, A note "On transformation preserving asymptotic lines of surfaces in Three-Dimensional Euclidean Space", Submitted).

In Alagöz and Soyuçok (2009), the transformation preserving the asymptotic directions of hypersurfaces in n-dimensional Euclidean space is considered and obtained a system of equation which must be satisfied by transformations.

In this paper, we consider the transformations which preserve the asymptotic directions of hypersurfaces in n-dimensional Euclidean space and we show that a trans-

formation preserves the asymptotic directions of hypersurfaces if and only if it is the projective one. Therefore we obtain a new characterization of the projective transformation.

In the n-dimensional Euclidean space, a hypersurface can be expressed by the equation;

$$\mathbf{r}(u^1, \dots, u^{n-1}) = (x^1(u^1, \dots, u^{n-1}), x^2(u^1, \dots, u^{n-1}), \dots, x^n(u^1, \dots, u^{n-1})) \quad (1)$$

Where the metric of the space is given by

$$ds^2 = dx^1^2 + dx^2^2 + \dots + dx^n^2. \quad (2)$$

We assume that $\mathbf{r}(u^1, u^2, \dots, u^{n-1})$ is a differentiable function of order 3 and the tangent vectors $\mathbf{r}_{,1}, \mathbf{r}_{,2}, \dots, \mathbf{r}_{,n-1}$ of the hypersurface are linearly independent where

$$\mathbf{r}_{,i} \equiv \frac{\partial \mathbf{r}}{\partial u^i}, \quad i = 1, 2, \dots, n-1. \quad (3)$$

The first and second fundamental forms of the hypersurface are;

$$I = g_{ij} du^i du^j, \quad II = L_{ij} du^i du^j, \quad (i, j = 1, 2, \dots, n-1) \quad (4)$$

Where;

$$g_{ij} = \mathbf{r}_{,i} \cdot \mathbf{r}_{,j} \quad (5)$$

*Corresponding author. E-mail: ygulluk@yildiz.edu.tr.

$$L_{ij} = \mathbf{r}_{,ij} \cdot \mathbf{N}, \quad \left(\mathbf{r}_{,ij} = \frac{\partial^2 \mathbf{r}}{\partial u^i \partial u^j} \right) \quad (6)$$

and \mathbf{N} is the unit normal vector of the hypersurface. The differential equation of the asymptotic directions of the hypersurface is given by

$$L_{ij} du^i du^j = 0 \quad (7)$$

(Aminov, 2001 p.44) and (Weatherburn, 1963, p.134).

For a matrix

$$\mathbf{A} = \begin{bmatrix} x_{,1}^1 & x_{,1}^2 & \dots & x_{,1}^n \\ x_{,2}^1 & x_{,2}^2 & \dots & x_{,2}^n \\ \vdots & \vdots & \dots & \vdots \\ x_{,n-1}^1 & x_{,n-1}^2 & \dots & x_{,n-1}^n \end{bmatrix}, \quad \left(x_{,i}^p = \frac{\partial x^p}{\partial u^i} \right). \quad (8)$$

L_{ij} can be easily obtained from

$$kL_{ij} = \det \begin{bmatrix} x_{,ij}^1 & x_{,i1}^1 & x_{,i2}^1 & \dots & x_{,i,n-1}^1 \\ x_{,ij}^2 & x_{,i1}^2 & x_{,i2}^2 & \dots & x_{,i,n-1}^2 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ x_{,ij}^n & x_{,i1}^n & x_{,i2}^n & \dots & x_{,i,n-1}^n \end{bmatrix}, \quad \left(x_{,ij}^p = \frac{\partial^2 x^p}{\partial u^i \partial u^j} \right). \quad (9)$$

Where; k , in which Δ_i is the determinant of the matrix which is obtained by omitting i^{th} column in the coefficients matrix \mathbf{A} , is given by

$$k = \sqrt{\Delta_1^2 + \Delta_2^2 + \dots + \Delta_n^2}. \quad (10)$$

Taken according to parameters

$$u^1 = x^1, u^2 = x^2, \dots, u^{n-1} = x^{n-1} \quad (11)$$

a hypersurface S is defined by the equation

$$\mathbf{r}(x^1, x^2, \dots, x^{n-1}) = (x^1, x^2, \dots, x^{n-1}, x^n(x^1, x^2, \dots, x^{n-1})). \quad (12)$$

The differential equation of the asymptotic directions of the hypersurface S , from (9) and (7), is obtained as;

$$x_{,ij}^n dx^i dx^j = 0, \quad \left(x_{,ij}^n = \frac{\partial^2 x^n}{\partial x^i \partial x^j}; i, j = 1, 2, \dots, n-1 \right). \quad (13)$$

In the n -dimensional Euclidean space, \mathbf{T} is the general coordinate transformation

$$\mathbf{T}: y^a = y^a(x^1, x^2, \dots, x^n), \quad (a = 1, 2, \dots, n). \quad (14)$$

We assume that \mathbf{T} is differentiable of order 3 and

$$\Delta = \det[\mathbf{T}_{,1} \quad \mathbf{T}_{,2} \quad \dots \quad \mathbf{T}_{,n}] = |\mathbf{T}_{,1} \quad \mathbf{T}_{,2} \quad \dots \quad \mathbf{T}_{,n}| \neq 0 \quad (15)$$

Where;

$$\mathbf{T}_{,b} = \begin{bmatrix} y_{,b}^1 \\ y_{,b}^2 \\ \vdots \\ y_{,b}^n \end{bmatrix}, \quad \left(y_{,b}^a = \frac{\partial y^a}{\partial x^b}; b = 1, 2, \dots, n \right). \quad (16)$$

By the transformation \mathbf{T} , the hypersurfaces S is transformed to the hypersurface S^* which is given by the equation

$$\mathbf{r}^*(x^1, x^2, \dots, x^{n-1}) = (y^1, y^2, \dots, y^n) \quad (17)$$

Where;

$$y^a = y^a(x^1, x^2, \dots, x^{n-1}, x^n(x^1, x^2, \dots, x^{n-1})).$$

The differential equation of the asymptotic directions of the hypersurface S^* is given by

$$L_{ij}^* dx^i dx^j = 0. \quad (18)$$

In order that the transformation \mathbf{T} transforms the asymptotic directions of the hypersurface S to the asymptotic directions of the hypersurface S^* , a system of equations

$$\mathbf{T}_{,ab} = A_b \mathbf{T}_{,a} + A_a \mathbf{T}_{,b}, \quad (19)$$

Is obtained where A_1, A_2, \dots, A_n arbitrary functions of variables are x^1, x^2, \dots, x^n . That is, this equation system must be satisfied by the transformation \mathbf{T} . Therefore, the following theorem can be written.

Theorem 1: A transformation \mathbf{T} which preserves the asymptotic directions of a hypersurface must satisfy the equations

$$\mathbf{T}_{,ab} = A_b \mathbf{T}_{,a} + A_a \mathbf{T}_{,b}, \quad (a, b = 1, 2, \dots, n) \quad (20)$$

Where; A_1, A_2, \dots, A_n are arbitrary functions of variables x^1, x^2, \dots, x^n , (Alagöz and Soyuçok, 2009).

A CHARACTERIZATION OF THE PROJECTIVE TRANSFORMATION

Firstly, let us consider the projective transformation

$$\mathbf{T}: y^a = \frac{C_0^a + C_1^a x^1 + L + C_n^a x^n}{C_0 + C_1 x^1 + L + C_n x^n} = \frac{C_p^a x^p}{C_p x^p} \quad (21)$$

($a = 1, 2, K, n$) that is,

$$\mathbf{T} = \frac{\mathbf{C}_p x^p}{C_p x^p}, \quad (\mathbf{C}_p = (C_p^1, C_p^2, K, C_p^n)) \quad (22)$$

Where; C_p^m and C_p are constants. For this transformation we get

$$\mathbf{T}_{,a} = \frac{(C_p C_a - C_a C_p) x^p}{(C_p x^p)^2}, \quad \mathbf{T}_{,b} = \frac{(C_p C_b - C_b C_p) x^p}{(C_p x^p)^2} \quad (23)$$

and

$$\mathbf{T}_{,ab} = \frac{-C_b (C_p C_a - C_a C_p) x^p - C_a (C_p C_b - C_b C_p) x^p}{(C_p x^p)^3}. \quad (24)$$

So we have

$$\mathbf{T}_{,ab} = \frac{-C_b}{C_p x^p} \mathbf{T}_{,a} + \frac{-C_a}{C_p x^p} \mathbf{T}_{,b}. \quad (25)$$

Accordingly, the projective transformation satisfies (19). Hence, according to the Theorem 1, the projective transformation preserves the asymptotic directions of a hypersurface.

Example: Let us consider, in 5-dimensional Euclidean space, the hypersurface S which is given by the equation

$$\mathbf{r} = (x^1, x^2, x^3, x^4, x^5(x^1, x^2, x^3, x^4)) \quad (26)$$

Where;

$$x^5 = (x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2. \quad (27)$$

Accordingly, the equation of the asymptotic directions of the S hypersurface, because of;

$$x_{11}^5 = 2, x_{22}^5 = 2, x_{33}^5 = -2, x_{44}^5 = -2, x_{ij}^5 = 0 \quad (i \neq j; i, j = 1, 2, 3, 4) \quad (28)$$

from (13) is found

$$(dx^1)^2 + (dx^2)^2 - (dx^3)^2 - (dx^4)^2 = 0. \quad (29)$$

See also Aminov (2001, p.45).

By the projective transformation in 5-dimensional Euclidean space, the hypersurface S transforms to the hypersurface S^* which is given by the equation

$$\mathbf{r}^* (x^1, x^2, x^3, x^4) = (y^1, y^2, y^3, y^4, y^5) \quad (30)$$

Where;

$$\begin{aligned} y^1 &= \frac{a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 ((x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2)}{c_0 + c_1 x^1 + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 ((x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2)}, \\ y^2 &= \frac{b_0 + b_1 x^1 + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 ((x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2)}{c_0 + c_1 x^1 + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 ((x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2)}, \\ y^3 &= \frac{d_0 + d_1 x^1 + d_2 x^2 + d_3 x^3 + d_4 x^4 + d_5 ((x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2)}{c_0 + c_1 x^1 + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 ((x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2)}, \\ y^4 &= \frac{e_0 + e_1 x^1 + e_2 x^2 + e_3 x^3 + e_4 x^4 + e_5 ((x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2)}{c_0 + c_1 x^1 + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 ((x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2)}, \\ y^5 &= \frac{f_0 + f_1 x^1 + f_2 x^2 + f_3 x^3 + f_4 x^4 + f_5 ((x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2)}{c_0 + c_1 x^1 + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 ((x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2)}. \quad (31) \end{aligned}$$

Then, from (9) and (10)

$$k^* L^*_{11} = \frac{B}{C}, \quad k^* L^*_{22} = \frac{B}{C}, \quad k^* L^*_{33} = \frac{-B}{C}, \quad k^* L^*_{44} = \frac{-B}{C}, \quad (32)$$

$$k^* L^*_{ij} = 0, \quad (i \neq j; i, j = 1, 2, 3, 4) \quad (33)$$

are found. Here

B is a constant which consists of the summation of 720 terms which are the multiplication of constants in the form $a_p b_q c_r d_s e_t f_v$ ($p, q, r, s, t, v = 0, 1, 2, 3, 4, 5$),

$$C = (c_0 + c_1 x^1 + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 ((x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2))^6 \quad (34)$$

$$\text{and } k^* = \sqrt{(\Delta_1^*)^2 + (\Delta_2^*)^2 + (\Delta_3^*)^2 + (\Delta_4^*)^2 + (\Delta_5^*)^2}.$$

Therefore, the equation of the asymptotic directions of the hypersurface S^* from (18), is found as

$$(dx^1)^2 + (dx^2)^2 - (dx^3)^2 - (dx^4)^2 = 0. \quad (35)$$

That is, the projective transformation preserves the asymptotic directions of given hypersurface.

In the following, we show that a transformation satisfying the conditions of the Theorem 1 is the projective one.

From the compatibility equations $T_{,abc} = T_{,acb}$ of (19) we obtain

$$(A_{b,c} - A_{c,b})T_{,a} + (A_{a,c} - A_{c,a})T_{,b} + (A_a A_b - A_{a,b})T_{,c} = 0 \quad (36)$$

From this we have

$$A_{a,b} = A_a A_b \quad (37)$$

and so,

$$A_{a,a} = A_a^2, \quad \left(A_a^2 = \frac{\partial A_a}{\partial x^a} \right). \quad (38)$$

Hence we have

$$A_a = \frac{1}{-x^a + B^a}, \quad (39)$$

Where; $B^a = B^a(x^1, x^2, K, x^{a-1}, x^{a+1}, K, x^n)$.

From (39), using (37) we get

$$B_{,b}^a = -\frac{-x^a + B^a}{-x^b + B^b} \quad (40)$$

which implies

$$B_{,bc}^a = 0. \quad (41)$$

Hence we obtain

$$B^a = d_0^a + d_1^a x^1 + d_2^a x^2 + L + d_{a-1}^a x^{a-1} + d_{a+1}^a x^{a+1} + L + d_n^a x^n \quad (42)$$

Where; d_p^a are constants. Thus we have

$$A_a = \frac{1}{-x^a + d_0^a + d_1^a x^1 + d_2^a x^2 + L + d_{a-1}^a x^{a-1} + d_{a+1}^a x^{a+1} + L + d_n^a x^n} \quad (43)$$

or

$$A_a = \frac{-C_a}{g}, \quad (44)$$

Where;

$$g = C_p x^p, \quad (p = 0, 1, 2, K, n) \quad (45)$$

and C_0, C_1, K, C_n are constants, with $C_a \neq 0$. By this value of A_a , from (19), which is written for $a = b$, we have

$$T_{,a} = \frac{f_a}{g^2} \quad (46)$$

Where; $f_a = f_a(x^1, x^2, K, x^{a-1}, x^{a+1}, K, x^n)$. Using this value of $T_{,a}$ in (19) we obtain

$$f_{a,b} = \frac{C_b f_a - C_a f_b}{g} \quad (47)$$

And by the differentiation we get

$$f_{a,bc} = 0. \quad (48)$$

Accordingly, we have

$$f_a = C_{aq} x^q \quad (49)$$

Where; C_{ab} are constant vectors and $C_{aa} = 0$. So

$$T_{,a} = \frac{C_{aq} x^q}{g^2}. \quad (50)$$

Using (50) in (19) we get

$$C_q C_{ab} = C_b C_{aq} - C_a C_{bq} \quad (51)$$

By the integration of (50) we find

$$T = -\frac{C_{aq} x^q}{C_a g} + h_a, \quad (52)$$

Where; $h_a = h_a(x^1, x^2, K, x^{a-1}, x^{a+1}, K, x^n)$. Because of (51)

$$h_{a,b} = 0 \quad (53)$$

That is, \mathbf{h}_a is a constant vector. So we have

$$\mathbf{T} = \frac{\mathbf{e}_p x^p}{C_p x^p} = \frac{\mathbf{e}_0 + \mathbf{e}_1 x^1 + L + \mathbf{e}_n x^n}{C_o + C_1 x^1 + L + C_n x^n} \quad (54)$$

Where; the vectors \mathbf{e}_p are constants. So our transformation is the projective transformation.

Therefore we have the following theorem which gives a new characterization of the projective transformation.

Theorem 2: A transformation preserves the asymptotic directions of a hypersurface if and only if it is the projective one.

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