

Full Length Research Paper

# Norm properties of operators whose norms are Eigenvalues

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Accepted 19 May, 2010

In this paper we present properties of a norm-attaining operator on a Hilbert space and there implications. We show that if  $T$  has norm attaining vector then  $\left\| \left( \sum_{k=0}^n \alpha_k T^k \right) x \right\| = \sum_{k=0}^n \alpha_k \|T\|^k$  where the scalars are nonnegative numbers. Thus  $T$  satisfies a generalized Daugavet condition.

**Key words:** Numerical range, eigenvalue, normaloid operator, Daugavet property.

## INTRODUCTION

In this article we extend results obtained by C.S Lin on properties of an operator whose norm is an Eigen value. It is noteworthy that each compact operator on a Hilbert space (Halmos, 1974) has a norm-attaining vector (Shilov, 1965). Thus, these properties are characteristics of compact operators. We will denote operators on a Hilbert space by capital letters. The numerical range of an operator  $T$  is the convex set of complex numbers defined by  $W(T) = \{(Tx, x) : \|x\| = 1, x \in H\}$ . We shall denote by  $T^*$  the ad joint of  $T$ . We say that  $T$  satisfy the Daugavet (1963) equation if  $\|I+T\| = 1 + \|T\|$ . A unit vector  $x$  in  $H$  is a norm attaining vector for  $T$  if  $\|T\| = \|Tx\|$ . Lin (2002) wrote a paper on a bounded operator on a Hilbert space whose norm is an eigenvalue and established the following theorem namely that

## THEORY

### Theorem 1

Let  $x$  be a unit vector in  $H$ . Then the following are equivalent:

1.  $\|T\|$  is an eigenvalue of  $T$ , that is,  $Tx = \|T\|x$ .

2.  $1 + \|T\| = \|(I+T)x\|$ .

3.  $\|T\|$  is an eigenvalue of  $T$  and  $Tx = \|Tx\|x$  that is,  $Tx = \|T\|x$  and  $Tx = \|Tx\|x$ .

4.  $\|T\|$  is in the numerical range of  $T$  that is,  $\|T\| = (Tx, x)$ .

5.  $x$  is a complete vector for  $T$ , that is,  $\|T\| = (Tx, x) = \|Tx\|$

6.  $2\|T\|$  is an eigen-value of  $T+T^*$ , that is  $(T+T^*)x = 2\|T\|x$ .

7.  $\|T\|$  and  $\|T\|^2$  are eigen-values of  $T$  and  $T^*T$ , respectively, with respect to  $x$ , that is,  $Tx = \|T\|x$  and  $T^*Tx = \|T\|^2x$ .

8.  $(1 + \|T\|)\|T\|$  is an eigen-value of  $(I+T^*)T$ , that is,  $(I+T^*)Tx = (1 + \|T\|)\|T\|x$ .

9.  $\|T\|$  is a normal eigenvalue for  $T$ , that is,  $Tx = \|T\|x = T^*x$ .

10.  $x$  is a complete vector for  $T$  and  $T^*$ , that is,  $Tx = (Tx, x) = \|Tx\| = \|T^*x\|$ .

11.  $x$  is a complete vector for  $T$  and

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$T^*T$  , that is,  $\|T\| = (Tx, x) = \|Tx\|$  and  $\|T\|^2 = \|Tx\|^2 = \|T^*Tx\|$

$$12. 1 + \|T\| + \|T\|^2 = \|(I + T + T^*T)x\|.$$

Now we make a natural extension of part 2 of theorem 1 as follows

**Lemma 2**

Let  $x$  be an operator on a Hilbert space  $H$  and  $x$  be a unit vector in  $H$ . Then the following statements are equivalent;

(i)  $x$  is an eigenvector of  $T$  with eigenvalue  $\|T\|$  , that is,  $Tx = \|T\|x$  .

(ii) For any sequence  $\alpha_1, \dots, \alpha_n$  of positive numbers

$$\left\| \left( \sum_{k=0}^n \alpha_k T^k \right) x \right\| = \sum_{k=0}^n \alpha_k \|T\|^k$$

**Proof.** (i)  $\rightarrow$  (ii) If  $\|T\|$  is an eigenvalue of  $T$  then it follows that

$$\sum_{k=0}^n \alpha_k \|T\|^{2k} = \left\| \left( \sum_{k=0}^n \alpha_k (T^*T)^k \right) x \right\|^2$$

$$\begin{aligned} \left\| \left( \sum_{k=0}^n \alpha_k T^k \right) x \right\|^2 &= \sum_{k=0}^n \alpha_k \|T\|^{2k} \\ \left\| \sum_{k=0}^n \alpha_k \|T\|^k x \right\|^2 &= \sum_{k=0}^n \alpha_k \|T\|^{2k} \|x\|^2 = \sum_{k=0}^n \alpha_k \|T\|^{2k} \end{aligned}$$

(ii)  $\rightarrow$  (i) Now set  $\alpha_k = 0, \alpha_1 = 1, k \neq 1$  to obtain  $\|Tx\| = \|T\|$  .

Also from  $\alpha_0 = \alpha_1 = 1, \alpha_k = 0, k > 1$  we obtain  $1 + \|T\| = \|(I + T)x\|$  .

$$\text{Hence } \|(I + T)x\|^2 = (1 + \|T\|)^2 .$$

Consequently,  $\|(I + T)x\|^2 = ((I + T)x, (I + T)x)$

$$\begin{aligned} &= (x, x) + (Tx, x) + (x, Tx) + (Tx, Tx) \\ &= 1 + (Tx, x) + (x, Tx) + \|T\|^2 = 1 + 2\|T\| + \|T\|^2 \end{aligned}$$

This leads to the result  $(Tx, x) + (x, Tx) = 2\|T\|$  . To show that  $\|T\|$  is an eigenvalue of  $T$  we consider the following expansion.

$$\begin{aligned} \|Tx - \|T\|x\|^2 &= (Tx - \|T\|x, Tx - \|T\|x) \\ &= (Tx, Tx) - \|T\|(Tx, x) - \|T\|(x, Tx) + \|T\|^2(x, x) \\ &= \|T\|^2 - \|T\|[(Tx, x) + (x, Tx)] + \|T\|^2 \\ &= \|T\|^2 - \|T\|(2\|T\|) + \|T\|^2 \\ &= 0 \end{aligned}$$

Hence  $Tx - \|T\|x = 0 \Leftrightarrow Tx = \|T\|x$  and so  $\|T\|$  is an eigenvalue of  $T$ .

In the same article Lin proved the theorem below which enumerates the properties of an operator  $a$  with norm attaining vector

**Theorem 3**

If  $T$  is an operator on a Hilbert space  $H$  and  $x$  is norm one vector in  $H$  then the following are equivalent statements; any of the statements in theorem 1

$$(ii) \left\| \left( \sum_{k=0}^n \alpha_k T^k \right) x \right\| = \sum_{k=0}^n \alpha_k \|T\|^k$$

Proof

It follows immediately from lemma 1

In the same article Lin proved the following

**Theorem 4**

Let  $x$  be a unit vector. Then the following are equivalent.

1.  $x$  is a norm attaining vector for  $T$  that is,  $\|T\| = \|Tx\|$  .
2.  $\|T\|^2$  is an eigenvalue for  $T$  , that is,  $T^*Tx = \|T\|^2 x$  .
3.  $1 + \|T\|^2 = \|(I + T^*T)x\|^2$  .

4.  $x$  is a complete vector

$$T^*T, \text{ that is, } \|T\|^2 = \|(T^*T)x\| = \|Tx\|^2$$

5.  $\|T\|^2$  is an eigenvalue for  $T^*T$ , and  $T^*Tx = \|(T^*T)x\|x$ ,

$$\text{that is, } T^*Tx = \|T\|^2 x \text{ and } T^*Tx = \|(T^*T)x\|x.$$

6.  $\|T\|^2$  is in the numerical range of  $T^*T$ , that is,

$$\|T\|^2 = ((T^*T)x, x).$$

$$7. 1 + \|T\|^2 + \|T\|^4 = \|I + T + T^*T + (T^*T)^2\|.$$

8.  $\|T\|^2$  and  $\|T\|^4$  are eigenvalues for  $T^*T$  and  $(T^*T)^2$ , respectively with respect to  $x$ , that is,

$$T^*Tx = \|T\|^2 x \text{ and } (T^*T)^2 x = \|T\|^4 x.$$

9.

$$1 + \|(I + T + T^*T)x\| = 1 + \|Tx\| + \|T^2x\| = \|(I + T + T^*T)x\| = \|(I + T + T^*T)x\| = \|(I + T + T^*T)x\|$$

$(1 + \|T\|^2)\|T\|^2$  is an eigenvalue of  $(I + T^*T)T^*T$ , that is,

$$(I + T^*T)T^*T x = (1 + \|T\|^2)\|T\|^2 x.$$

10.  $x$  is a complete vector for  $T^*T$  and  $(T^*T)^2$ , that is,

$$\|T\|^2 = \|Tx\|^2 = \|T^*Tx\| \text{ and } \|T\|^4 = \|T^*Tx\|^2 = \|(T^*T)^2 x\|.$$

We now prove a general result to the above in the following lemma

**Lemma 5**

Let  $T$  be an operator on a Hilbert space  $H$  and let  $x$  be a unit vector in  $H$  then the following are equivalent statements:

(i)  $x$  is an eigenvector of  $T^*T$  with Eigen value  $\|T\|^2$  that

$$\text{is, } T^*Tx = \|T\|^2 x$$

(ii) For any sequence  $\alpha_1, \dots, \alpha_n$  of positive numbers

$$\sum_{k=0}^n \alpha_k \|T\|^{2k} = \left\| \left( \sum_{k=0}^n \alpha_k (T^*T)^k \right) x \right\|$$

**Proof**

If we replace  $T$  with  $T^*T$  then we obtain the (i) if and only if

$$\sum_{k=0}^n \alpha_k \|T^*T\|^k = \left\| \left( \sum_{k=0}^n \alpha_k (T^*T)^k \right) x \right\|.$$

But we have that  $\|T^*T\| = \|T\|^2$ . Hence we obtain the result.

**Theorem 6**

Let  $T$  be an operator on a Hilbert space  $H$  and  $x$  be a unit vector then the following are equivalent statements; Any statement in theorem 6

For any  $\alpha_1, \dots, \alpha_n$  positive numbers

$$\sum_{k=0}^n \alpha_k \|T\|^{2k} = \left\| \left( \sum_{k=0}^n \alpha_k (T^*T)^k \right) x \right\|$$

**Proof**

The result follows from lemma

The following corollary which shows that if  $\|T\|$  is an eigenvalue of  $T$  with respect to  $x$ , then  $x$  is a norm attaining vector for  $T$  and satisfies the Daugavet property that is,

**Corollary 7**

Let  $x$  be a unit vector. Then any statement in theorem 2 implies the following;

Any statement in theorem 4

$T$  satisfy the Daugavet equation, that is,  $1 + \|T\| = \|(I + T)x\|$ .

$T$  and  $T^*$  satisfy the generalized Daugavet equation  $\|(I + T + T^*)\| = 1 + 2\|T\|$ .

$T$  and  $T^*T$  satisfy the generalized Daugavet equation  $\|(I + T + T^*T)\| = 1 + \|T\| + \|T\|^2$ .

$T$  is a normaloid operator, that is,  $r(T) = \|T\|$ .

$x$  is a norm attaining vector for  $I + T$ , that is,  $\|I + T\| = \|(I + T)x\|$ .

$x$  is a norm attaining vector for  $I + T + T^*$ , that is,  $\|I + T + T^*\| = \|(I + T + T^*)x\|$ .

$x$  is a norm attaining vector for  $I + T + T^*T$ , that is,  $\|I + T + T^*T\| = \|(I + T + T^*T)x\|$

**Proof**

As in Lin with Theorem 1 and 2 now replaced with 3 and 4. We now consider further results when the operator  $T$  is

both self ad joint and compact. In this case the operator has a norm attaining vector as shown by Shilov

**Theorem 8**

If T is a compact self ad joint operator then  $\|T^n\| = w(T^n) = (w(T))^n = \|T\|^n$ .

**Proof**

Since T is self ad joint  $T^n$  is also self ad joint and so the first equality follows from corollary

2. Also, if x is the norm attaining vector for T we have ;

$$(T^n x; x) = (T^{n-1} x; T x) = \|T\| (T^{n-1} x; x) = \|T\| (T^{n-2} x; T x) = \|T\|^2 (T^{n-2} x; x) = \|T\|^n = (w(T))^n.$$

But we have  $w(T^n) \geq (T^n x; x)$ . Consequently  $w(T^n) \geq (w(T))^n$ .

For the reverse inequality we note that  $w(T^n) = \|T^n\| \leq \|T\|^n = (w(T))^n$  corollary 9

If T is a compact self adjoint operator then

$$w\left(\sum_{k=0}^n \alpha_k T^k\right) = \left\| \sum_{k=0}^n \alpha_k T^k \right\| = \sum_{k=0}^n \alpha_k \|T\|^k = \sum_{k=0}^n \alpha_k (w(T))^k$$

Where  $\alpha_k$  are non negative numbers.

**Proof**

The first equality follows from the fact that the sum of self ad joint operators is also self ad joint. The second follows from properties of norm attaining operators. A compact self-ad joint operator therefore satisfies a generalized Daugavet equation (Lin, 2002).

**Conclusion**

The main result here is that if an operator satisfies a Daugavet condition then it also satisfies a generalized Daugavet condition with nonnegative scalars, that is,

$$\left\| \left( \sum_{k=0}^n \alpha_k T^k \right) x \right\| = \sum_{k=0}^n \alpha_k \|T\|^k$$

. It would be of interest if this result can be extended to an infinite sum.

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